Ministry of Education and Science of Ukraine
Dniprovsky State Technical University

Dmytro Redchyts

SPECIAL COURSES OF APPLIED MATHEMATICS

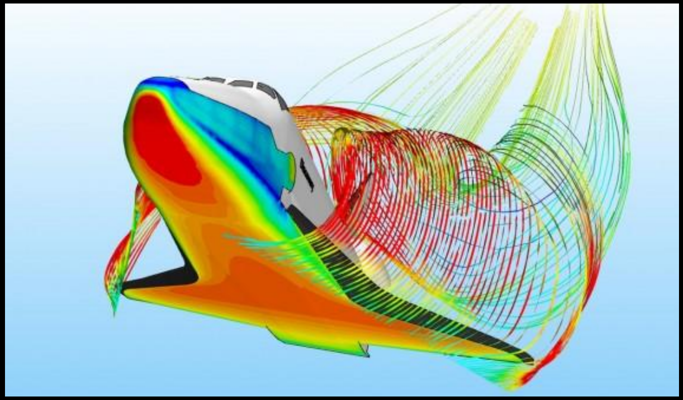
specialty 113 “Applied Mathematics”

Lecture notes 3

BASIC APPROACHES TO SOLVING PROBLEMS IN MATHEMATICAL PHYSICS

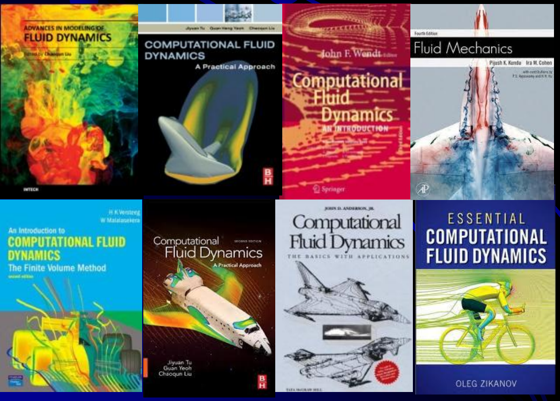
Kamianske, Ukraine
10 October 2025

Methodology of applied mathematics



Experiment, physics phenomena

$$\iiint_V \frac{\partial q}{\partial t} dV + \iint_S \left(E - E_v \right) dS = 0$$



Mathematical models

Computer technology

The role of numerical methods in applied mathematics

Initial differential equations in continuous space from mathematical models

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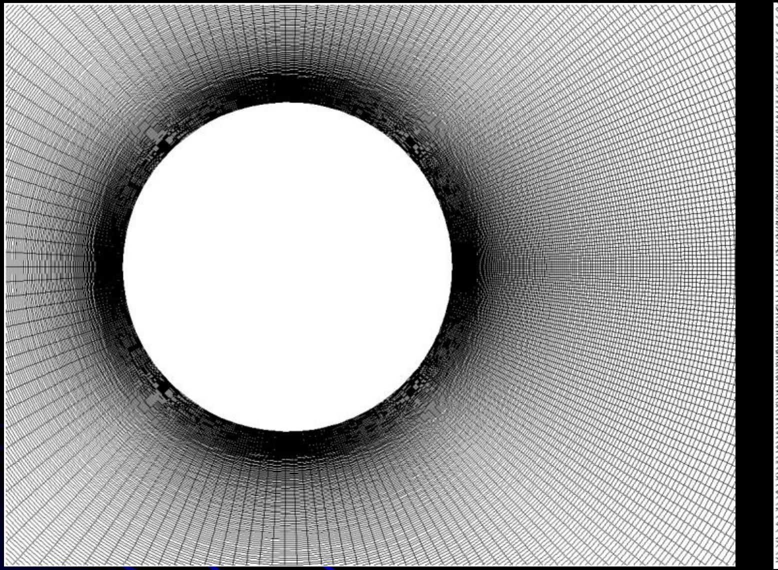
NUMERICAL METHODS

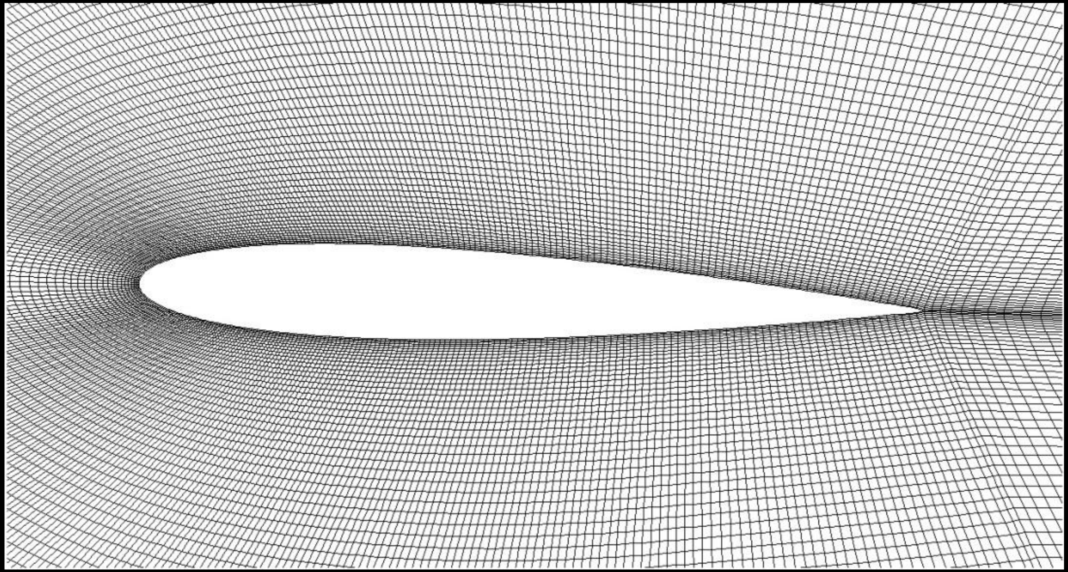
↓

Discrete analogs of the initial differential equations in discrete space for computer technologies

Creation the discrete space



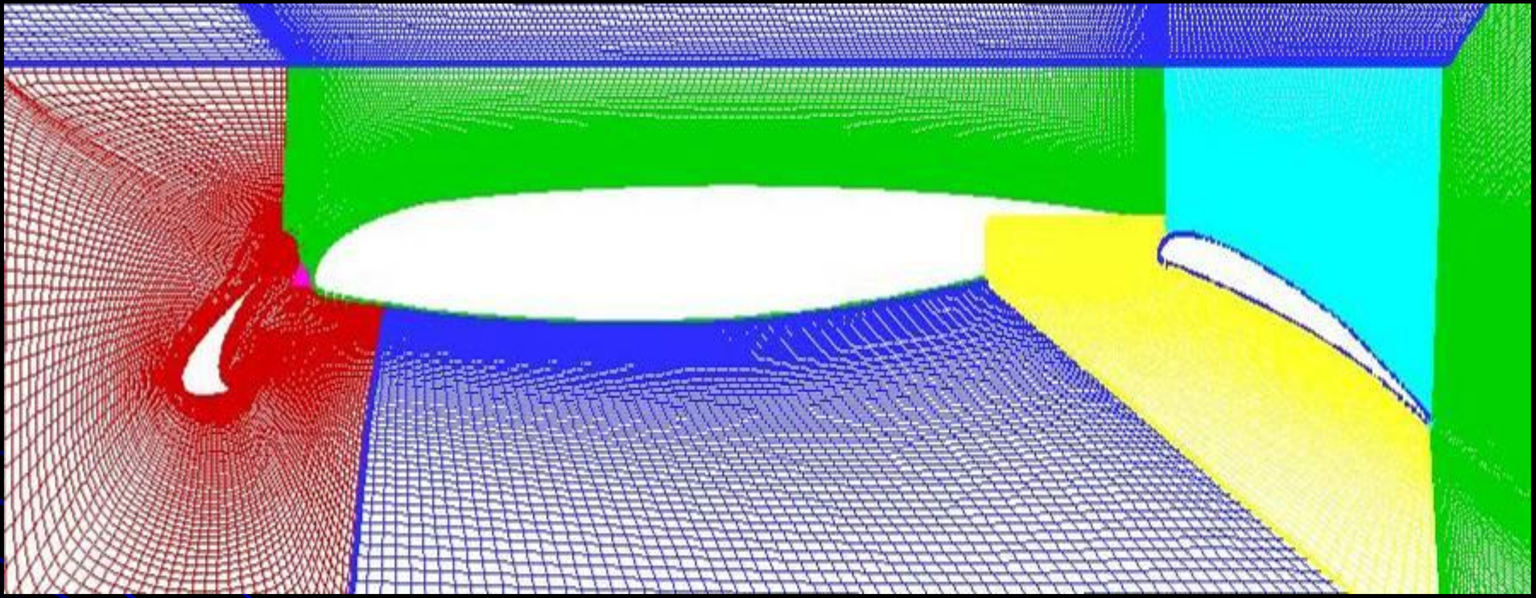
Circular cylinder



NACA 0012 airfoil

Structured grids

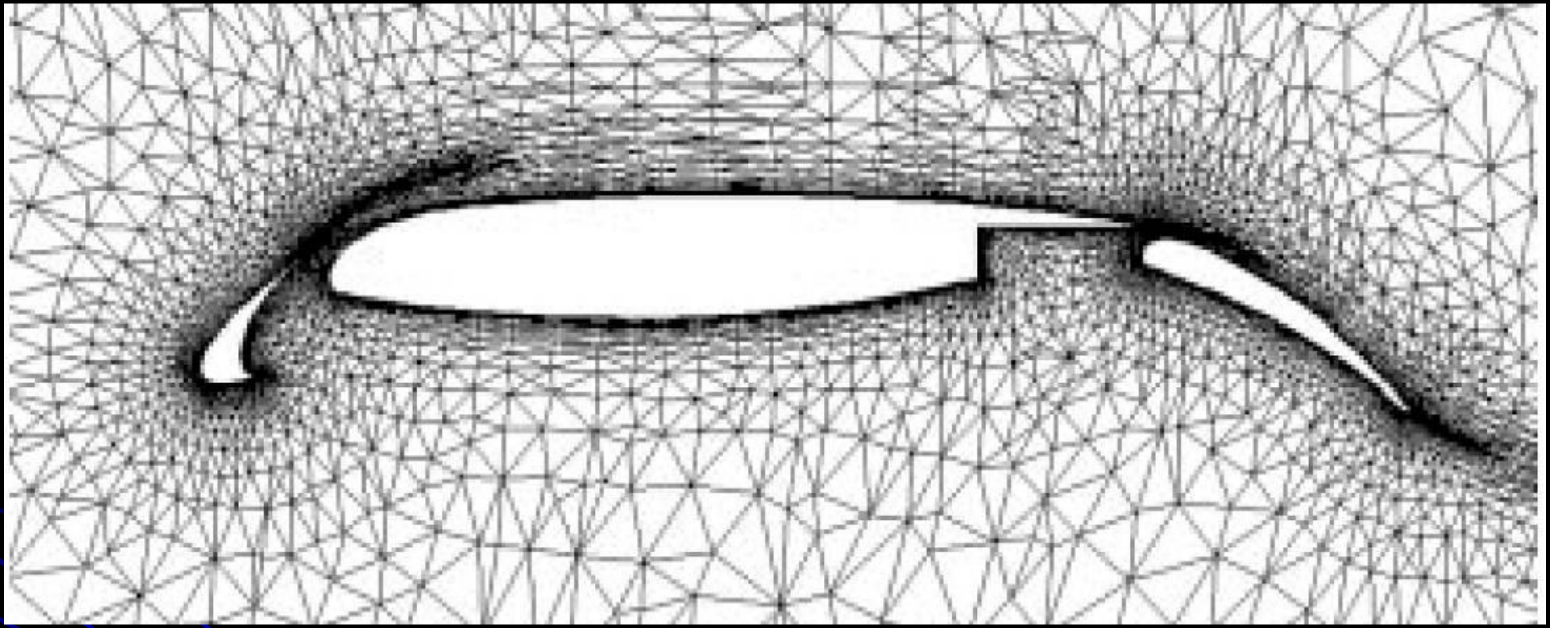
Creation the discrete space



Three-element airfoil 30P30N

Structured multi-block grid

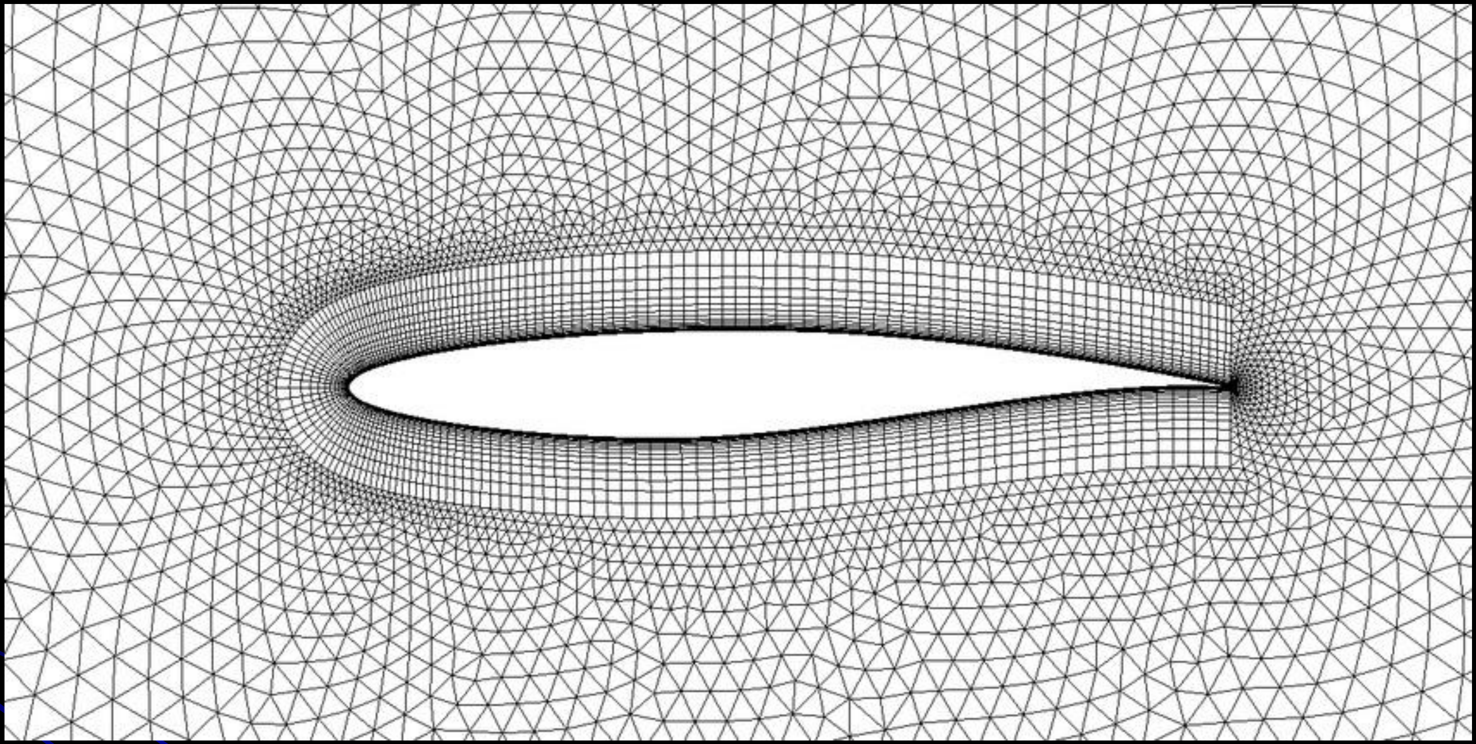
Creation the discrete space



Three-element airfoil 30P30N

Unstructured grid

Creation the discrete space



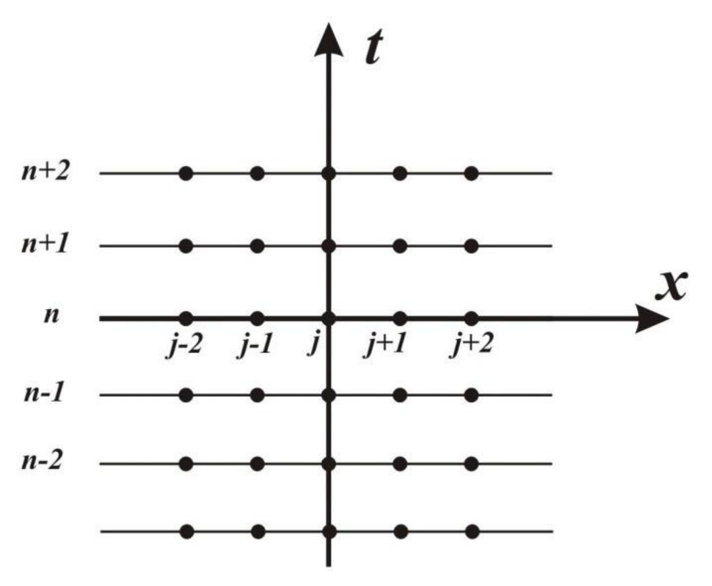
NACA 4412 airfoil

Hybrid structured and unstructured grid

Creation discrete analogs of initial differential equations in discrete space

$$\frac{\partial \overline{T}}{\partial t} = a\frac{\partial^2 \overline{T}}{\partial x^2}$$

One-dimensional heat equation



Discretization in space and time

Creation discrete analogs of initial differential equations in discrete space

Layout in Taylor series around the point ***x* = *j* ·Δ*x*, *t* = *n* ·Δ*t***

$$\overline{T}_{j+1}^{n}=\sum_{m=0}^{\infty}\frac{\Delta x^{m}}{m!}\left[\frac{\partial^{m}\overline{T}}{\partial x^{m}}\right]_{j}^{n}\qquad\qquad \overline{T}_{j}^{n+1}=\sum_{m=0}^{\infty}\frac{\Delta t^{m}}{m!}\left[\frac{\partial^{m}\overline{T}}{\partial t^{m}}\right]_{j}^{n}$$

or

$$\overline{T}_{j}^{n+1}=\overline{T}_{j}^{n}+\Delta t\left[\frac{\partial\overline{T}}{\partial t}\right]_{j}^{n}+\frac{\Delta t^{2}}{2}\left[\frac{\partial^{2}\overline{T}}{\partial t^{2}}\right]_{j}^{n}+\frac{\Delta t^{3}}{6}\left[\frac{\partial^{3}\overline{T}}{\partial t^{3}}\right]_{j}^{n}+\frac{\Delta t^{4}}{24}\left[\frac{\partial^{4}\overline{T}}{\partial t^{4}}\right]_{j}^{n}+O\left(\Delta t^{5}\right)$$

$$\overline{T}_{j+1}^{n}=\overline{T}_{j}^{n}+\Delta x\left[\frac{\partial\overline{T}}{\partial x}\right]_{j}^{n}+\frac{\Delta x^{2}}{2}\left[\frac{\partial^{2}\overline{T}}{\partial x^{2}}\right]_{j}^{n}+\frac{\Delta x^{3}}{6}\left[\frac{\partial^{3}\overline{T}}{\partial x^{3}}\right]_{j}^{n}+\frac{\Delta x^{4}}{24}\left[\frac{\partial^{4}\overline{T}}{\partial x^{4}}\right]_{j}^{n}+O\left(\Delta x^{5}\right)$$

Creation discrete analogs of initial differential equations in discrete space

First derivative in time

Two-layer approximation

$$\begin{cases} T_j^{n+1} = \overline{T}_j^n + \Delta t \left[\dfrac{\partial \overline{T}}{\partial t} \right]_j^n + O\left(\Delta t^2\right) & \rightarrow \times \left(\dfrac{1}{\Delta t} \right) \\ T_j^n = \overline{T}_j^n & \rightarrow \times \left(\dfrac{-1}{\Delta t} \right) \end{cases}$$

First order approximation

$$\left[\frac{\partial \overline{T}}{\partial t} \right]_j^n = \frac{T_j^{n+1} - T_j^n}{\Delta t} + O(\Delta t)$$

Creation discrete analogs of initial differential equations in discrete space

First derivative in time

Asymmetric three-layer approximation

$$\begin{cases} T_j^{n+1} = \bar{T}_j^{n+1} & \rightarrow \times\left(\frac{3}{2\Delta t}\right) \\ T_j^{n} = \bar{T}_j^{n+1} - \Delta t\left[\frac{\partial \bar{T}}{\partial t}\right]_j^{n+1} + \frac{\Delta t^2}{2}\left[\frac{\partial^2 \bar{T}}{\partial t^2}\right]_j^{n+1} + O\left(\Delta t^3\right) & \rightarrow \times\left(\frac{-4}{2\Delta t}\right) \\ T_j^{n-1} = \bar{T}_j^{n+1} - 2\Delta t\left[\frac{\partial \bar{T}}{\partial t}\right]_j^{n+1} + \frac{4\Delta t^2}{2}\left[\frac{\partial^2 \bar{T}}{\partial t^2}\right]_j^{n+1} + O\left(\Delta t^3\right) & \rightarrow \times\left(\frac{1}{2\Delta t}\right) \end{cases}$$

Second order approximation

$$\left[\frac{\partial \bar{T}}{\partial t}\right]_j^{n+1} = \frac{3T_j^{n+1} - 4T_j^{n} + T_j^{n-1}}{2\Delta t} + O(\Delta t^2)$$

Creation discrete analogs of initial differential equations in discrete space

Second derivative in space

Symmetric three-point approximation

$$
\begin{cases}
T_{j+1}^{n} = \overline{T}_{j}^{n} + \Delta x\left[\dfrac{\partial \overline{T}}{\partial x}\right]_{j}^{n} + \dfrac{\Delta x^{2}}{2}\left[\dfrac{\partial^{2} \overline{T}}{\partial x^{2}}\right]_{j}^{n} + \dfrac{\Delta x^{3}}{6}\left[\dfrac{\partial^{3} \overline{T}}{\partial x^{3}}\right]_{j}^{n} + \dfrac{\Delta x^{4}}{24}\left[\dfrac{\partial^{4} \overline{T}}{\partial x^{4}}\right]_{j}^{n} + O\left(\Delta x^{5}\right) & \rightarrow \times\left(\dfrac{1}{\Delta x^{2}}\right) \\
T_{j}^{n} = \overline{T}_{j}^{n} & \rightarrow \times\left(\dfrac{-2}{\Delta x^{2}}\right) \\
T_{j-1}^{n} = \overline{T}_{j}^{n} - \Delta x\left[\dfrac{\partial \overline{T}}{\partial x}\right]_{j}^{n} + \dfrac{\Delta x^{2}}{2}\left[\dfrac{\partial^{2} \overline{T}}{\partial x^{2}}\right]_{j}^{n} - \dfrac{\Delta x^{3}}{6}\left[\dfrac{\partial^{3} \overline{T}}{\partial x^{3}}\right]_{j}^{n} + \dfrac{\Delta x^{4}}{24}\left[\dfrac{\partial^{4} \overline{T}}{\partial x^{4}}\right]_{j}^{n} + O\left(\Delta x^{5}\right) & \rightarrow \times\left(\dfrac{1}{\Delta x^{2}}\right)
\end{cases}
$$

Second order approximation

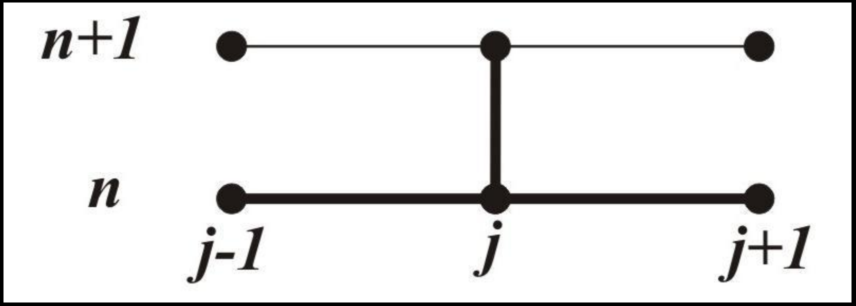
$$
\left[\frac{\partial^{2} \overline{T}}{\partial x^{2}}\right]_{j}^{n} = \frac{T_{j+1}^{n} - 2T_{j}^{n} + T_{j-1}^{n}}{\Delta x^{2}} + O(\Delta x^{2})
$$

Creation discrete analogs of initial differential equations in discrete space

Explicit scheme

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} = a\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\Delta x^2}$$

Stencil



Formula for calculations

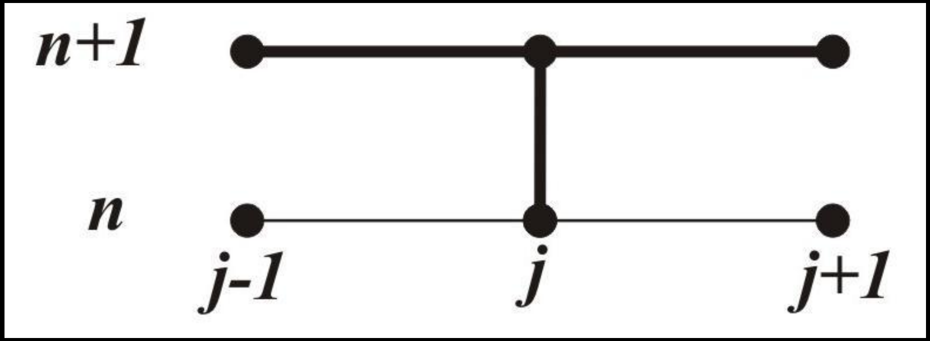
$$T_j^{n+1} = T_j^n + \frac{a\Delta t}{\Delta x^2}\left(T_{j+1}^n - 2T_j^n + T_{j-1}^n\right)$$

Creation discrete analogs of initial differential equations in discrete space

Implicit scheme

$$\frac{T_j^{n+1}-T_j^n}{\Delta t}=a\frac{T_{j+1}^{n+1}-2T_j^{n+1}+T_{j-1}^{n+1}}{\Delta x^2}$$

Stencil



Algebraic system of equations for calculations

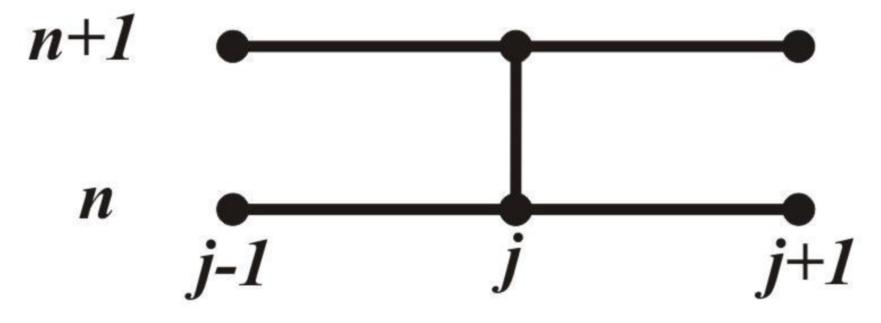
$$-\frac{a\Delta t}{\Delta x^2}T_{j+1}^{n+1}+\left(1+\frac{2a\Delta t}{\Delta x^2}\right)T_j^{n+1}-\frac{a\Delta t}{\Delta x^2}T_{j-1}^{n+1}=T_j^n$$

Creation discrete analogs of initial differential equations in discrete space

Δ-form of implicit scheme

$$\Delta T_j^n = T_j^{n+1} - T_j^n \qquad \frac{\Delta T_j^n}{\Delta t} = a\frac{\Delta T_{j+1}^{n+1} - 2\Delta T_j^{n+1} + \Delta T_{j-1}^{n+1}}{\Delta x^2} + a\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\Delta x^2}$$

Stencil



Algebraic system of equations for calculations

$$-\frac{a\Delta t}{\Delta x^2}\Delta T_{j+1}^{n+1} + \left(1 + \frac{2a\Delta t}{\Delta x^2}\right)\Delta T_j^{n+1} - \frac{a\Delta t}{\Delta x^2}\Delta T_{j-1}^{n+1} = \frac{a\Delta t}{\Delta x^2}\left(T_{j+1}^n - 2T_j^n + T_{j-1}^n\right)$$

Accuracy of the discretization process

$\bar{T} = \exp x$, step size is $\Delta x = 0{,}1$

Table 1 Comparison of formulas for calculation $d\bar{T} / dx$ at $x = 1{,}0$

Formula variant	Algebraic formula	$\left[\frac{d\bar{T}}{dx}\right]_j$	Error	Leading term of Taylor series
Accurate	–	2,7183	–	–
Three-point symmetrical	$\left(\bar{T}_{j+1} - \bar{T}_{j-1}\right) / 2\Delta x$	2,7228	$0{,}4533 \times 10^{-2}$	$0{,}4531 \times 10^{-2}$
Differences ahead	$\left(\bar{T}_{j+1} - \bar{T}_{j}\right) / \Delta x$	2,8588	$0{,}1406 \times 10^{-0}$	$0{,}1359 \times 10^{-0}$
Differences back	$\left(\bar{T}_{j} - \bar{T}_{j-1}\right) / \Delta x$	2,5868	$-0{,}1315 \times 10^{-0}$	$-0{,}1359 \times 10^{-0}$
Three-point asymmetric	$\left(-1{,}5\bar{T}_{j} + 2\bar{T}_{j+1} - 0{,}5\bar{T}_{j+2}\right) / \Delta x$	2,7085	$-0{,}9773 \times 10^{-2}$	$-0{,}9061 \times 10^{-2}$
Five-point symmetrical	$\left(\bar{T}_{j-2} - 8\bar{T}_{j-1} + 8\bar{T}_{j+1} - \bar{T}_{j+2}\right) / 12\Delta x$	2,7183	$-0{,}9072 \times 10^{-5}$	$-0{,}9061 \times 10^{-5}$

Accuracy of the discretization process

$\overline{T} = \exp x$, step size is $\Delta x = 0{,}1$

Table 2 Comparison of formulas for calculation $d^2\overline{T} / dx^2$ at $x = 1{,}0$

Formula variant	Algebraic formula	$\left[\frac{d^2\overline{T}}{dx^2}\right]$	Errors	Leading term of Taylor series
Accurate	–	$2{,}7183$	–	–
Three-point symmetrical	$\left(\overline{T}_{j-1} - 2\overline{T}_j + \overline{T}_{j+1}\right) / \Delta x^2$	$2{,}7205$	$0{,}2266 \times 10^{-2}$	$0{,}2265 \times 10^{-2}$
Three-point asymmetric	$\left(\overline{T}_j - 2\overline{T}_{j+1} + \overline{T}_{j+2}\right) / \Delta x^2$	$3{,}0067$	$0{,}2884 \times 10^{-0}$	$0{,}2718 \times 10^{-0}$
Five-point symmetrical	$(-\overline{T}_{j-2} + 16\overline{T}_{j-1} - 30\overline{T}_j + 16\overline{T}_{j+1} - $ $-\overline{T}_{j+2}\big) / 12\Delta x^2$	$2{,}7183$	$-0{,}3023 \times 10^{-5}$	$-0{,}3020 \times 10^{-5}$

Accuracy of the discretization process

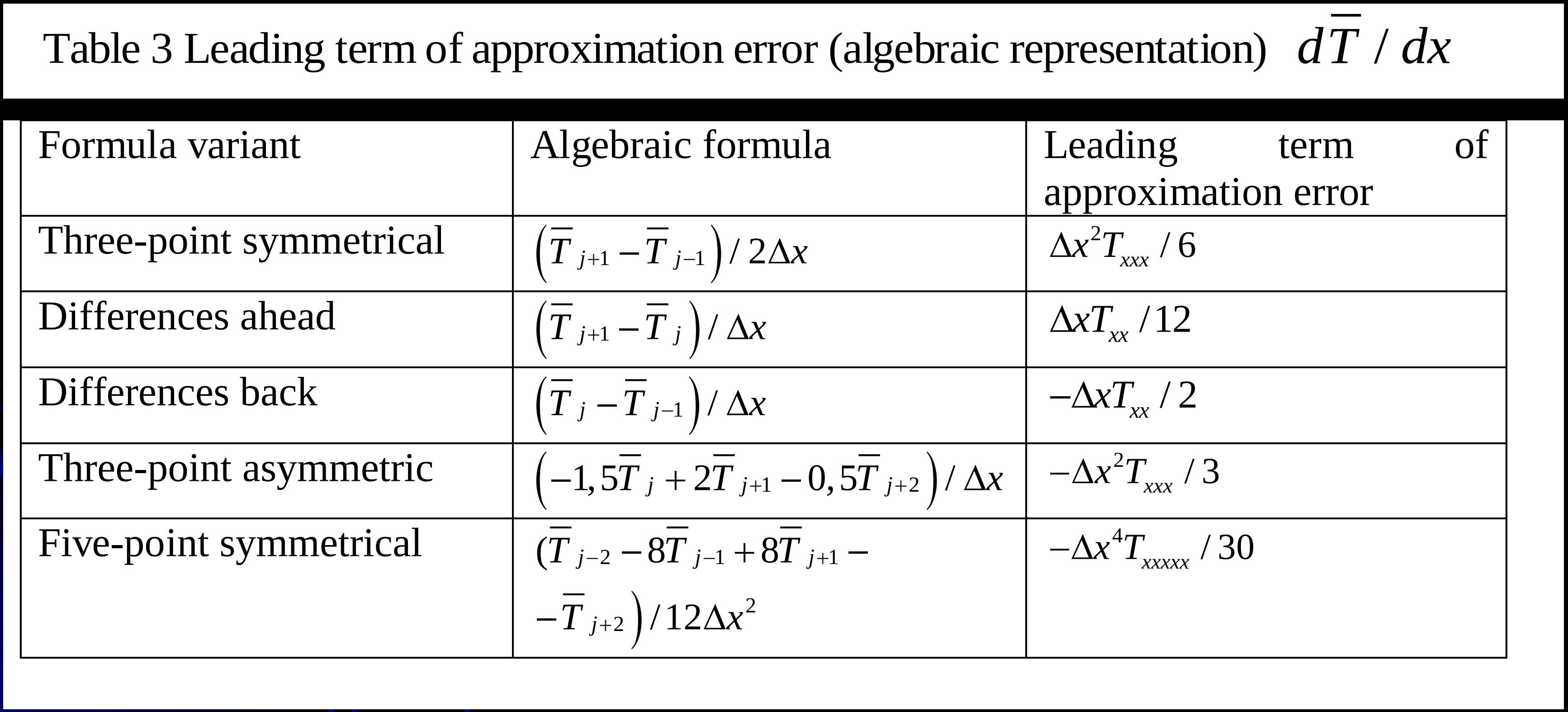


Table 3 Leading term of approximation error (algebraic representation) $d\overline{T} / dx$

Formula variant	Algebraic formula	Leading term of approximation error
Three-point symmetrical	$\left(\overline{T}_{j+1} - \overline{T}_{j-1}\right) / 2\Delta x$	$\Delta x^2 T_{xxx} / 6$
Differences ahead	$\left(\overline{T}_{j+1} - \overline{T}_{j}\right) / \Delta x$	$\Delta x T_{xx} / 12$
Differences back	$\left(\overline{T}_{j} - \overline{T}_{j-1}\right) / \Delta x$	$-\Delta x T_{xx} / 2$
Three-point asymmetric	$\left(-1{,}5\overline{T}_{j} + 2\overline{T}_{j+1} - 0{,}5\overline{T}_{j+2}\right) / \Delta x$	$-\Delta x^2 T_{xxx} / 3$
Five-point symmetrical	$(\overline{T}_{j-2} - 8\overline{T}_{j-1} + 8\overline{T}_{j+1} -$ $-\overline{T}_{j+2}\big) / 12\Delta x^2$	$-\Delta x^4 T_{xxxxx} / 30$

Accuracy of the discretization process

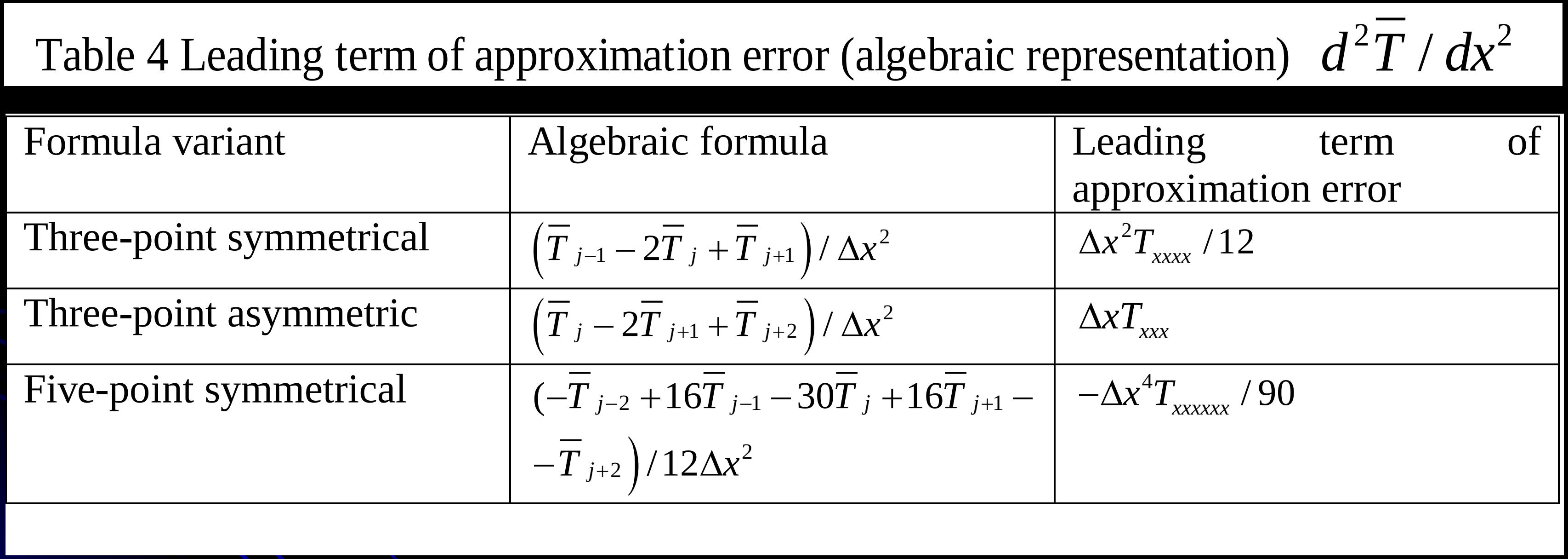


Table 4 Leading term of approximation error (algebraic representation) $d^2\overline{T}/dx^2$

Formula variant	Algebraic formula	Leading term of approximation error
Three-point symmetrical	$\left(\overline{T}_{j-1}-2\overline{T}_{j}+\overline{T}_{j+1}\right)/\Delta x^2$	$\Delta x^2 T_{xxxx}/12$
Three-point asymmetric	$\left(\overline{T}_{j}-2\overline{T}_{j+1}+\overline{T}_{j+2}\right)/\Delta x^2$	$\Delta x T_{xxx}$
Five-point symmetrical	$(-\overline{T}_{j-2}+16\overline{T}_{j-1}-30\overline{T}_{j}+16\overline{T}_{j+1}-$ $-\overline{T}_{j+2}\big)/12\Delta x^2$	$-\Delta x^4 T_{xxxxxx}/90$

Consistency of differential schemes

The system of algebraic equations obtained as a result of the discretization process ***is consistent*** with the initial partial differential equations (PDEs); when the grid step sizes approach zero, the system of algebraic equations is equivalent to the PDEs at each of the grid nodes.

It is clear that the presence of ***consistency*** is necessary if the approximate solution is to converge to the solution of the PDEs under consideration. However, this condition is not sufficient, since the system of algebraic equations will be equivalent to the PDEs when the grid step size approaches zero, hence it cannot be concluded that the solution of this system of algebraic equations will approach the solution of the original PDEs.

Consistent difference scheme is one that approximates a partial differential equation. Recall that the approximation error is the difference between a differential equation and its finite-difference analogue, so the condition for consistency of a difference scheme is that the approximation error tends to zero as the grid is refined. This condition is certainly satisfied if the approximation error decreases as the grid is refined, i.e. if the approximation error has order $\mathrm{O}(\Delta t), \mathrm{O}(\Delta x)$, etc.

Rounding error

Any numerically obtained solution, even a so-called exact analytical solution of a partial differential equation, depends on rounding errors associated with the finite number of signs used in arithmetic operations.

The error that occurs in this case is called ***rounding error***. It can significantly affect the solution of finite-difference equations, since obtaining this solution is usually associated with performing a large number of the same type of arithmetic operations. In some cases, the rounding error is proportional to the number of nodes of a different grid, so refining the grid, while reducing the approximation error, can increase the rounding error.

The approximation error is equal to the difference between the exact (excluding rounding errors) solutions of the original differential equation and its finite-difference analogue. Therefore, the error of the solution of the partial differential equation obtained on the computer is equal to the sum of the approximation and rounding errors. The accuracy of the numerical solution of the partial differential equation is determined by the approximation error of both the equation itself and the boundary conditions.

Stability of finite-difference schemes

If, when solving a system of algebraic equations, some disturbances arise that are not subject to control (rounding errors), and such disturbances tend to decay, then the solution will be ***stable***. Conversely, an unstable solution will give an amplitude of oscillations that increases with time. Therefore, the concept of stability is associated with the growth or decay of errors that are introduced into the calculation at any stage. These are precisely those errors that are associated with rounding directly in the process of calculation by a computer. A specific method and solution are considered ***stable*** if the cumulative effect of all rounding errors is negligibly small.

Similarly, the ***stability*** of a finite-difference scheme is determined by the change in error during the calculation. O'Brien et al. [O'Brien et al., 1950] proposed the following classification of the stability of difference schemes:

1. If the total rounding error increases (does not increase), then the difference scheme is called highly unstable (stable).
2. If a separate rounding error increases (does not increase), then the difference scheme is called weakly unstable (stable).

Usually, only weak stability is studied, since for its analysis it is possible to use the method of Fourier series expansion of the solution, which in terms of computational mathematics is called the Neumann method. It is assumed that if the condition of weak stability is met, then the condition of strong stability is met.

Convergence

The solution of algebraic equations that approximate a given PDEs is called ***convergent*** if this approximate solution of the PDEs approaches the exact solution of the PDEs for any value of the independent variable as the grid step size approaches zero.

Thus, we require that $T_j^n \to \bar{T}\left(x_j, t_n\right)$ when $\Delta x \to 0, \Delta t \to 0$.

The difference between the exact solution of a partial differential equation and the exact solution of a system of algebraic equations is called the solution error and is denoted by the symbol e_j^n

$$e_j^n = \bar{T}\left(x_j, t_n\right) - T_j^n$$

The exact solution of a system of algebraic equations is an approximate solution of the original partial differential equation. The solution of a system of algebraic equations is exact if no numerical errors, such as rounding errors, were introduced into it during the calculation process.

As a rule, the magnitude of the error e_j^n at $\left(j, n\right)$-th node depends on the sizes of the grid cells Δx and Δt, as well as on the values of those higher derivatives at this node that were not taken into account when formulating finite-difference approximations of the derivatives in the considered differential equation.

Computational efficiency

The question of what accuracy can be achieved with a given solution algorithm is related to the computational efficiency of that particular algorithm. Computational efficiency can be defined as the accuracy achieved per unit of execution time. Thus, an algorithm that achieves moderate accuracy on a coarse grid and takes a short time to execute may be as efficient as another algorithm that achieves high accuracy on a fine grid and takes a longer time to execute.

Computational efficiency (CE) can be determined by the formula, $CE = k/(\varepsilon ET)$ where ε is the error of the approximate solution, given in the corresponding norm (for example, in the root mean square), is the execution time.

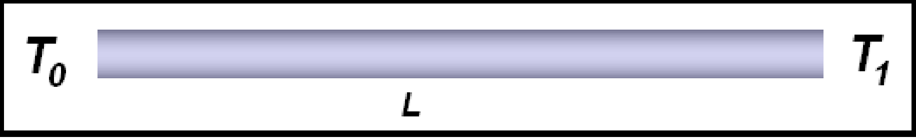
Computational efficiency is a useful criterion that allows us to clarify the feasibility of introducing high-order finite-difference schemes or higher-order finite elements.

For most thermal physics and hydroaerodynamics problems, the use of high-order (e.g., fourth-order or higher) finite-difference schemes or the use of higher-order finite elements (e.g., cubic or higher) is not justified in terms of increased computational efficiency, especially for three-dimensional problems.

When developing different computational schemes, it is preferable to compare them in terms of computational efficiency, and not only in terms of accuracy or only in terms of economy. If computational schemes are compared that are calculated on the same computer, then the execution time can be measured directly. If it is not possible to set the CPU time, attention should be paid to the operating mode of the computer.

Test calculations

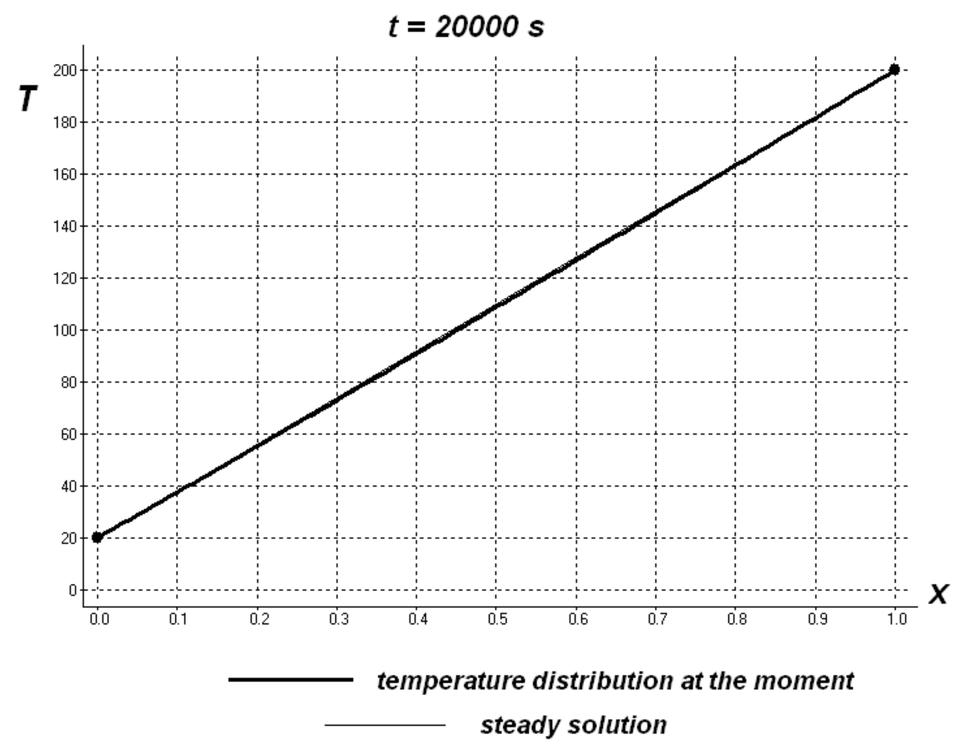
Heating an iron rod



$$\frac{\partial \overline{T}}{\partial t} = a\frac{\partial^2 \overline{T}}{\partial x^2}$$

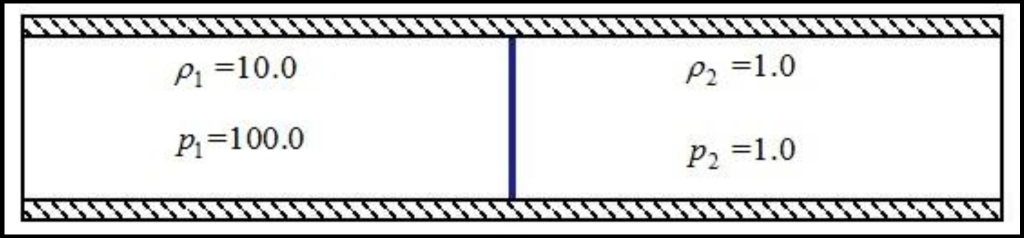
$T_0=20^0C$ $T_1=200^0C$ $L=1\ m$ $a=2.3\cdot10^{-5}\ m^2/s$

$$T(x) = \frac{T_1 - T_0}{L}x + T_0$$ - *steady solution*

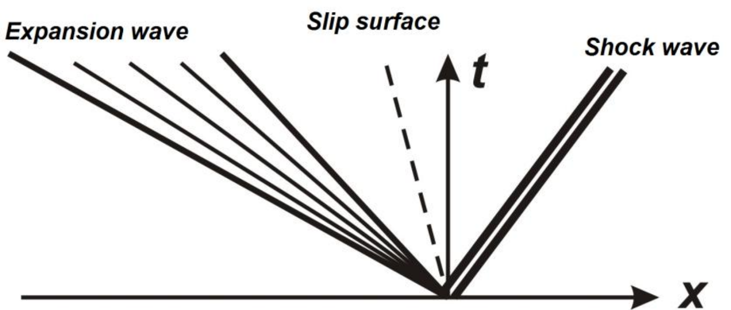


Test calculations

Shock Tube



Test NASA https://www.grc.nasa.gov/www/wind/valid/stube/stube.html



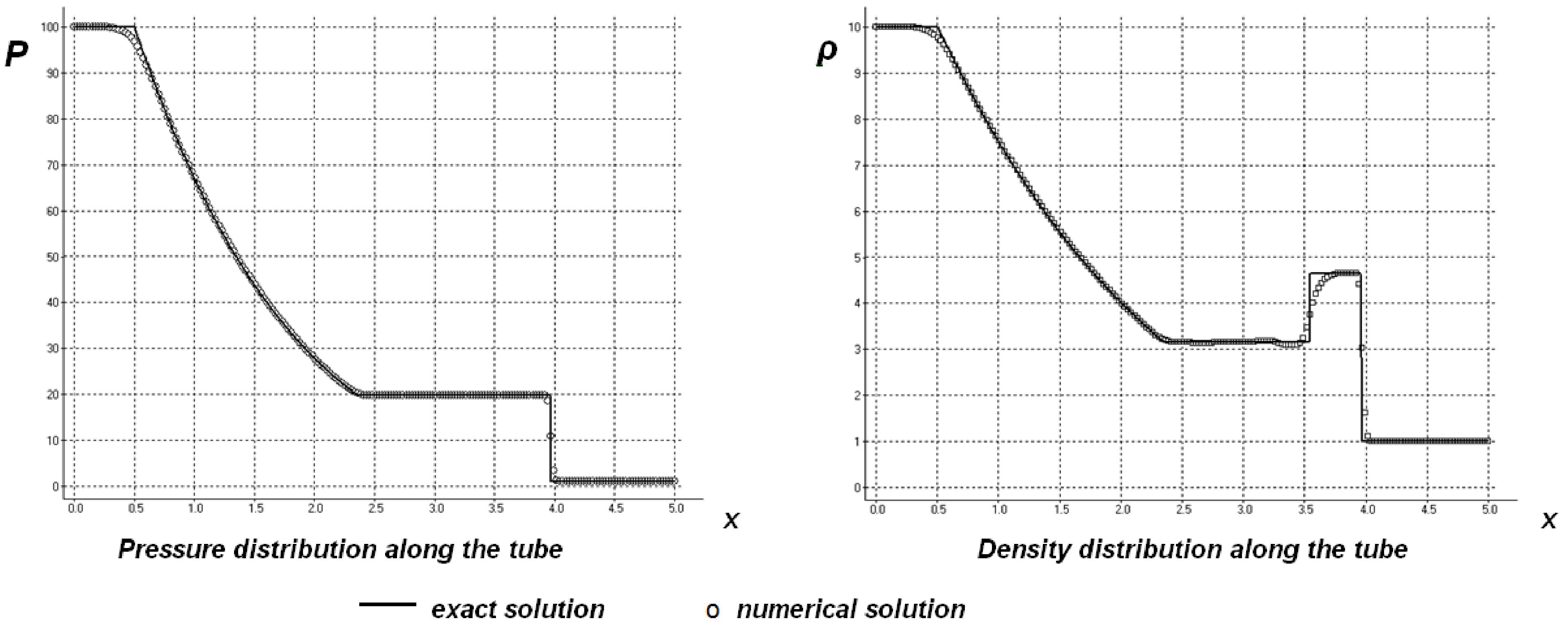
Shock tube shortly after diaphragm has burst

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \qquad \mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(e+p) \end{bmatrix}$$

One-dimensional Euler equations for compressible gas dynamics

Test calculations

Shock Tube



Pressure distribution along the tube

Density distribution along the tube

Comparison of exact and numerical solutions

Classification of software packages of computational fluid dynamics

	CFD packages	Developers	Users	Purpose
1	Specialized scientific and industry packages of large corporations	Centers – NASA, DLR, ONERA, NLR Company – Boeing, Lockheed Martin	CFD-experts	Real parameters of the flow
2	Commercial	ANSYS, FLUENT, StarCD, FLOW-3D, CFX, SolidWorks, ACE-U, FlowVision	Engineers, designers	Estimate of the flow parameters
3	Scientific and research	Universities and research centers	CFD-scientist	Research of flow physics
4	Individual	Students, postgraduate students	Students, postgraduate students	Learning of CFD

Scientific and research CFD codes

Preprocessor

- Formation of initial geometry
- Creation of discrete space
- Motion of individual blocks

Solver

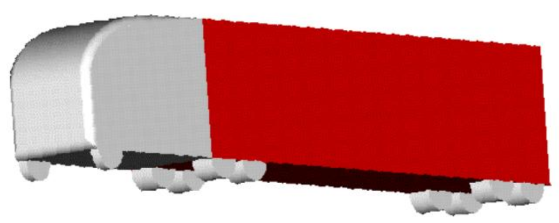
- Integration of Navier-Stokes equations
- Calculation of turbulence parameters
- Execution of boundary conditions
- Solution of coupled problems

Postprocessor

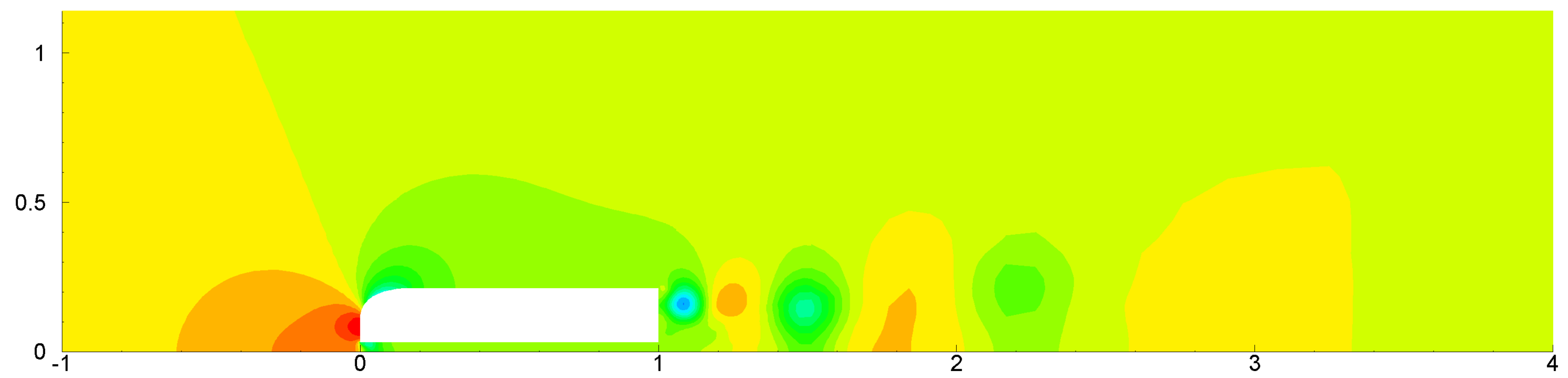
- Built-in visualization of results
- Calculation of aerodynamic characteristics
- Data preparation for external viewers

The integrated scheme of the CFD software developed by the authors

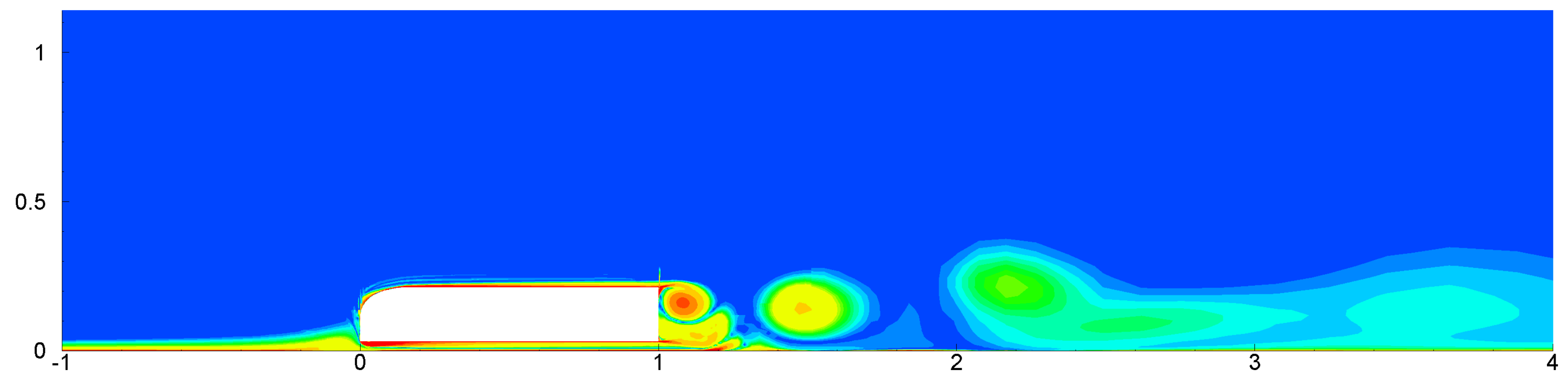
Aerodynamics of ground transport



Truck model GST Sandia

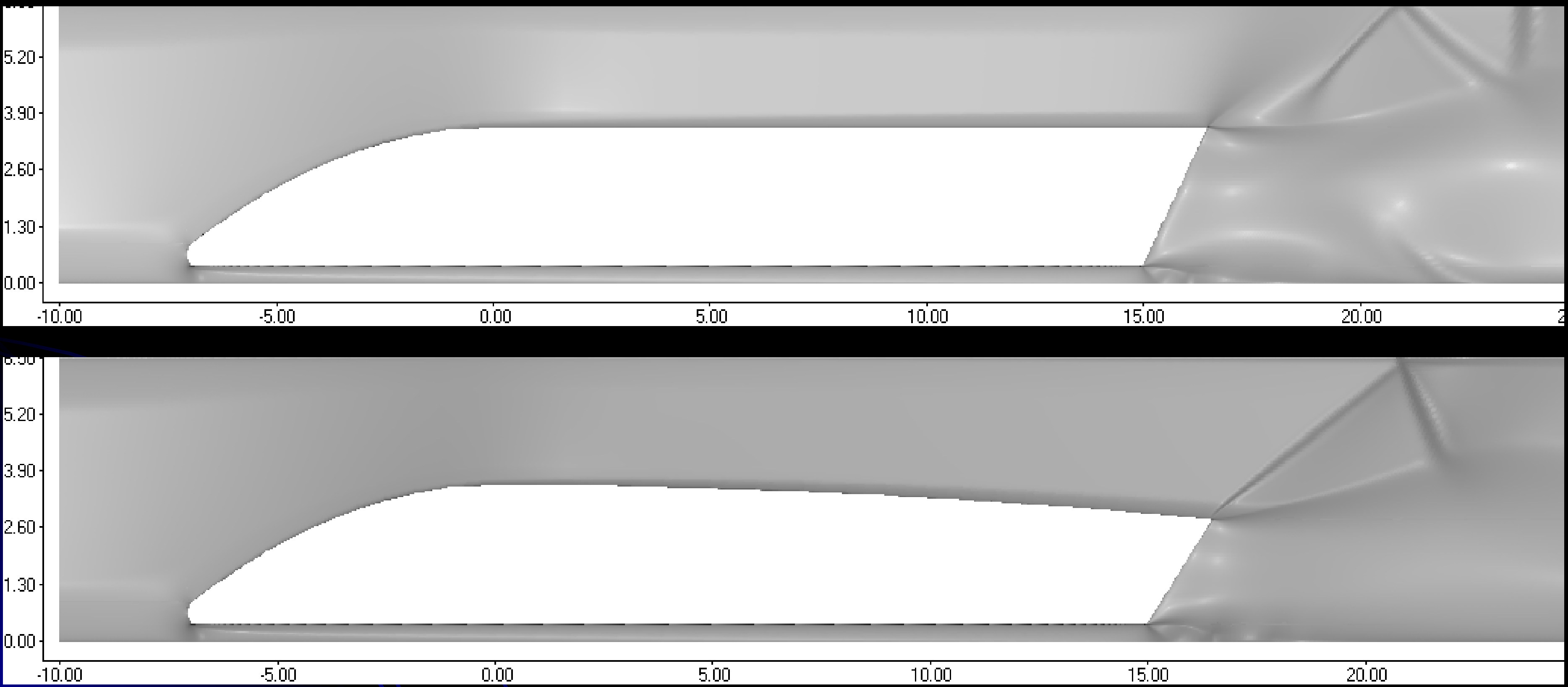


Pressure contours

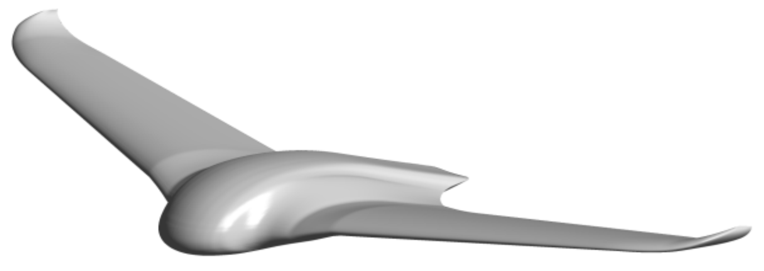


Vorticity magnitude contours

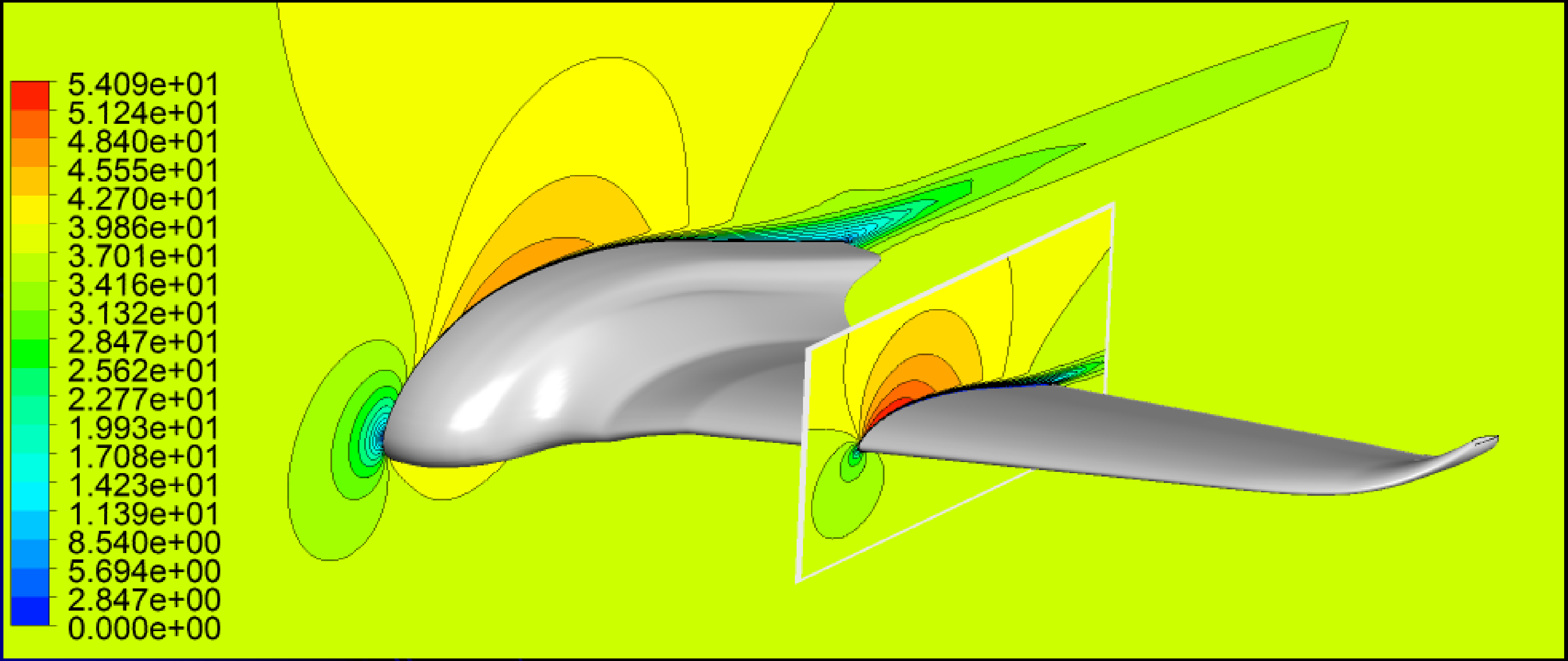
Aerodynamics of Hyperloop pod



Numerical “Schlieren photographs" of the flow near the HYPERLOOP pod of various geometry at a speed of 1080 km/h (300 m/s) and a pressure in the vacuum tube 101 Pa (0.001 atm)

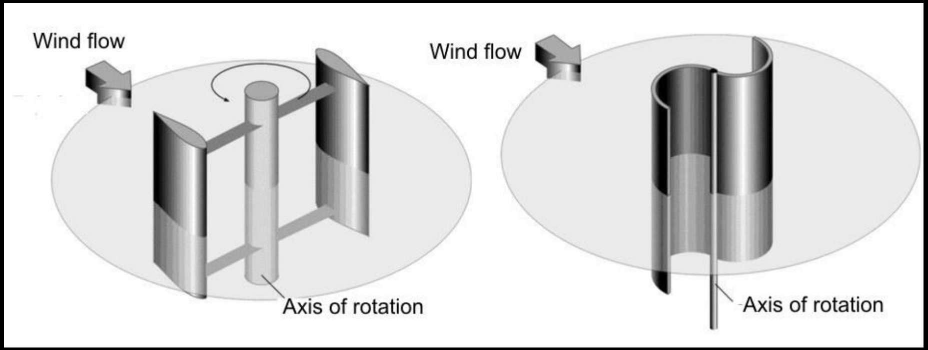


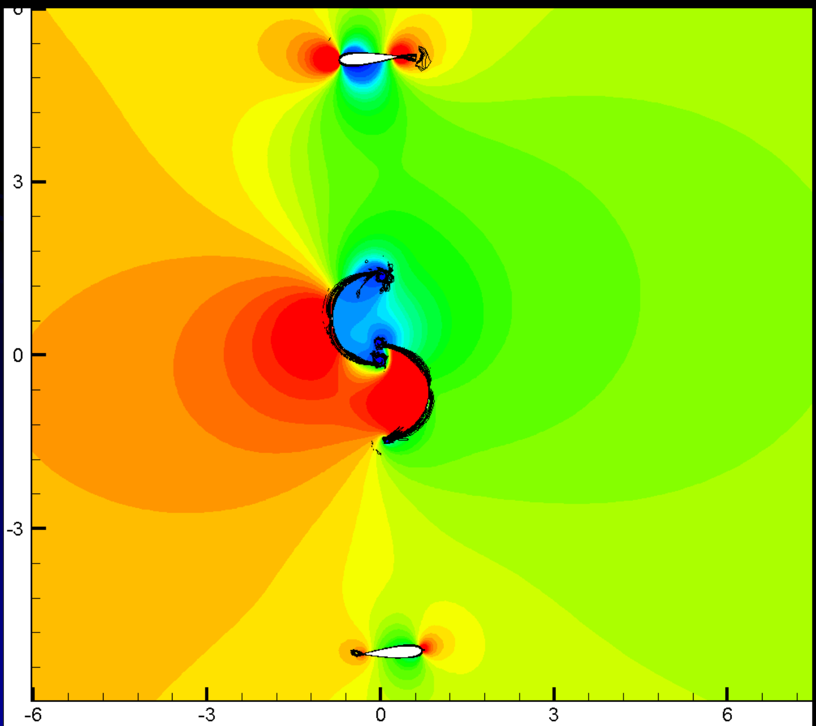
Aerodynamics of flying wing drone



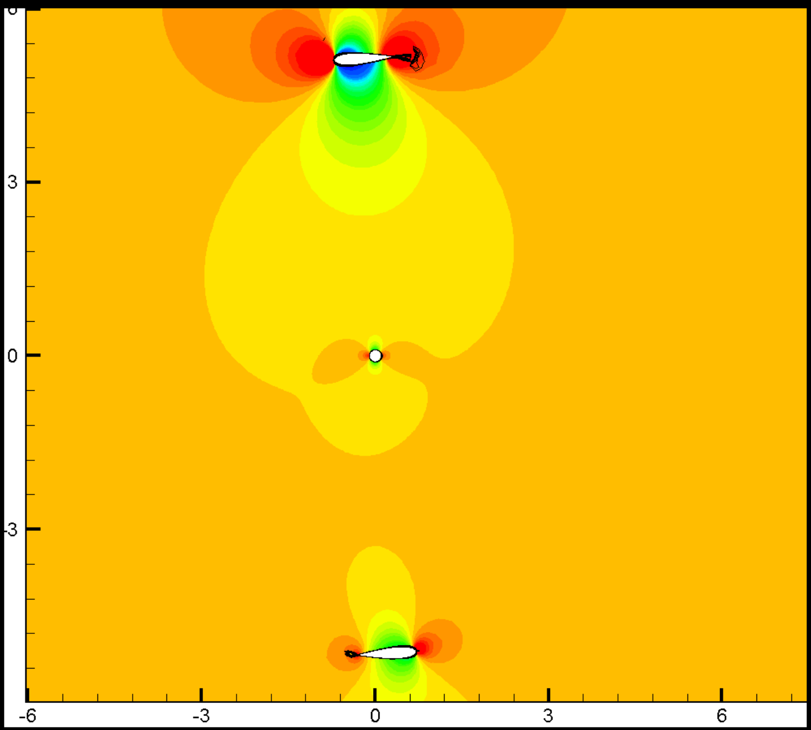
Velocity modulus distribution in longitudinal sections

Aerodynamics of vertical-axis wind turbines



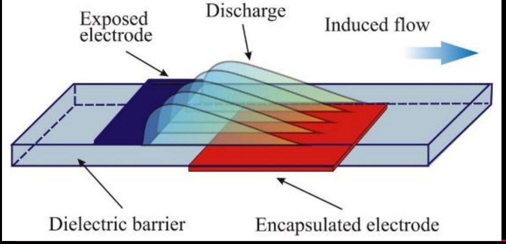


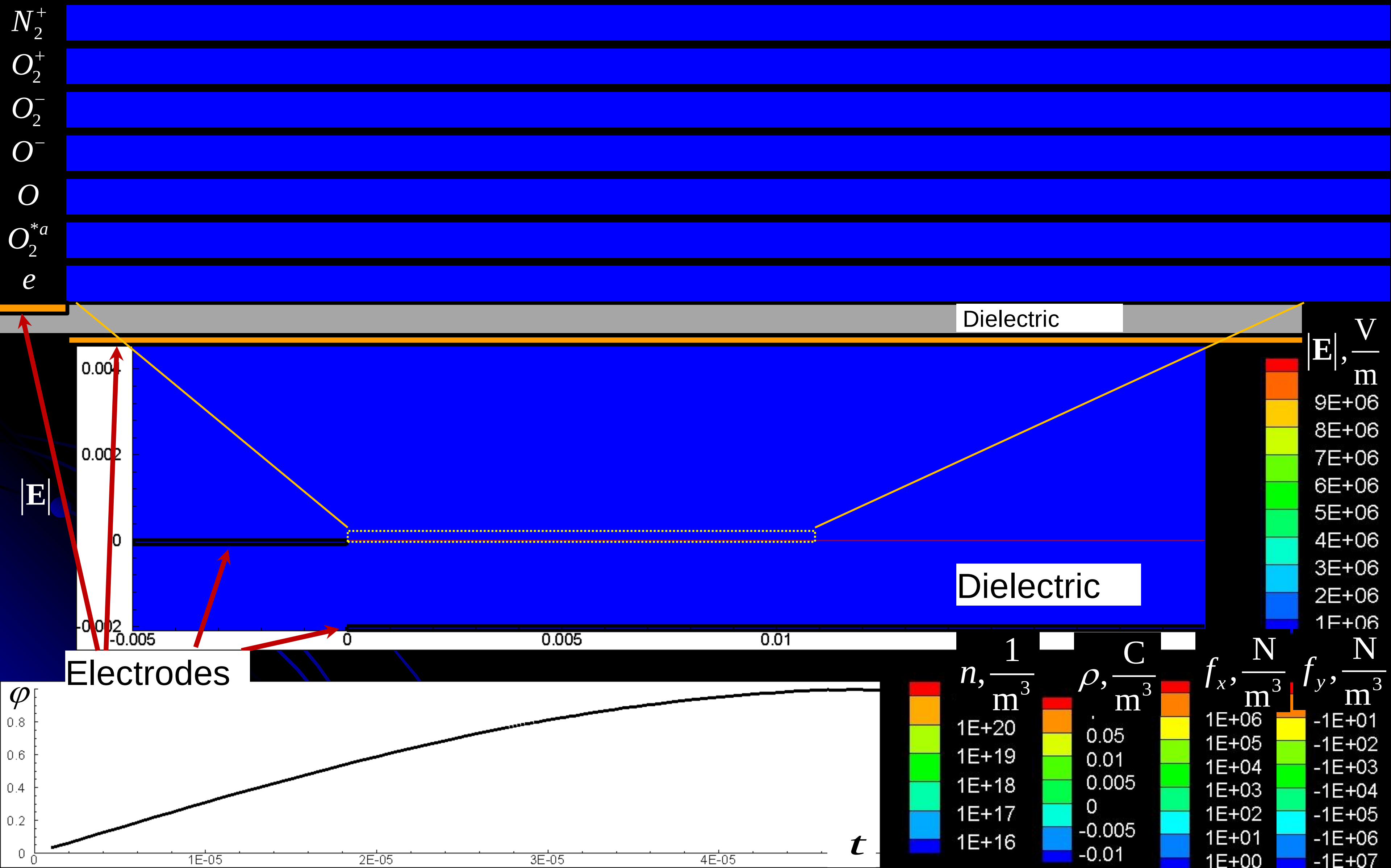
Joined Darrieus & Savonius rotors

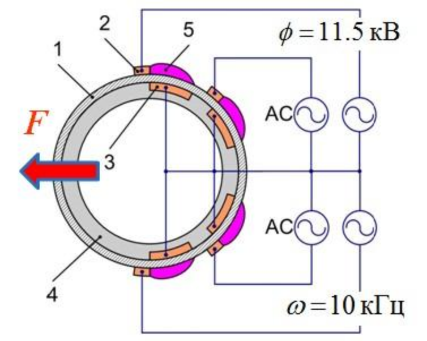


Darrieus rotor

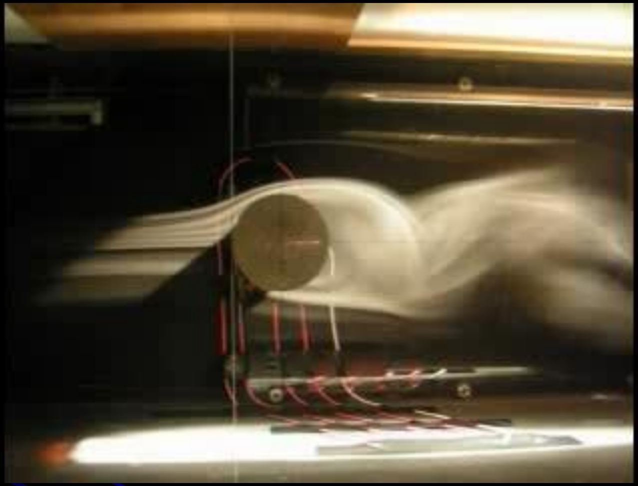
Plasma particle dynamics of dielectric barrier discharge



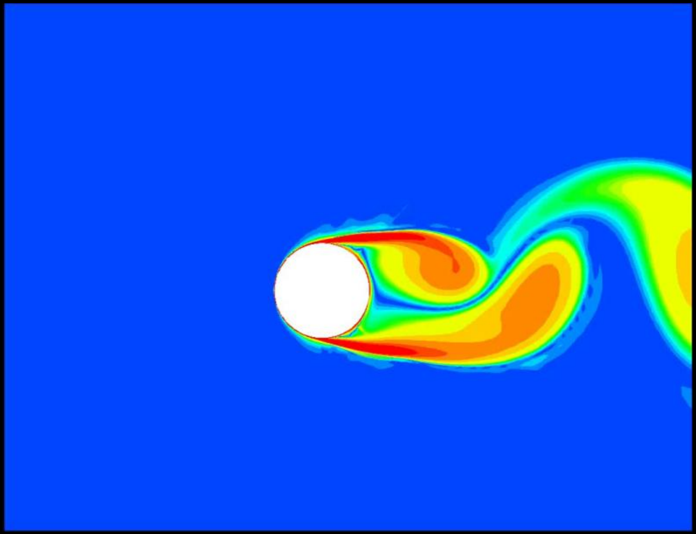




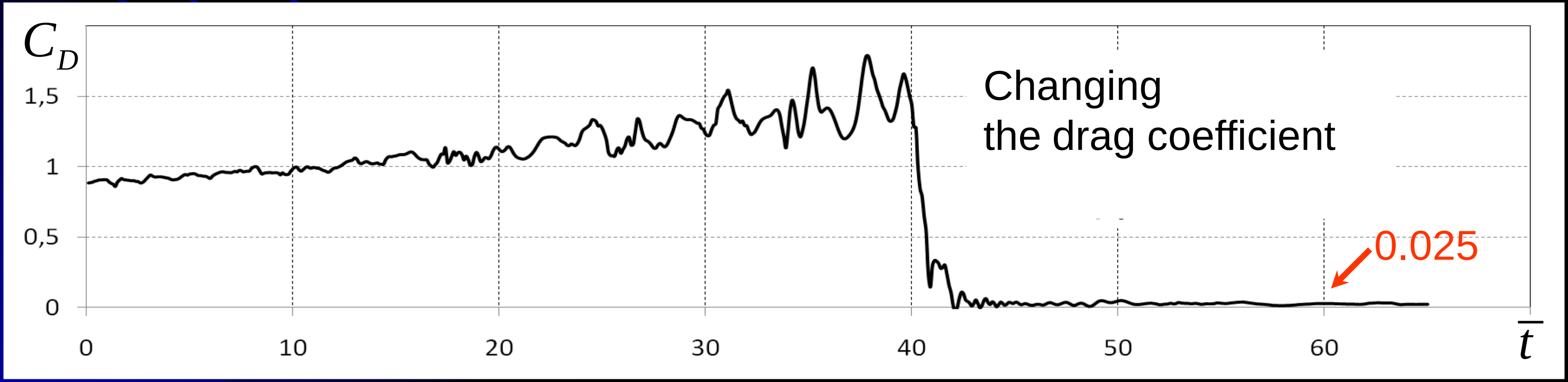
Flow separation control on a cylinder using 4 plasma actuators based on dielectric barrier discharge



Experiment



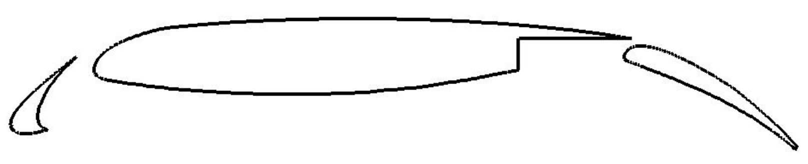
Calculation



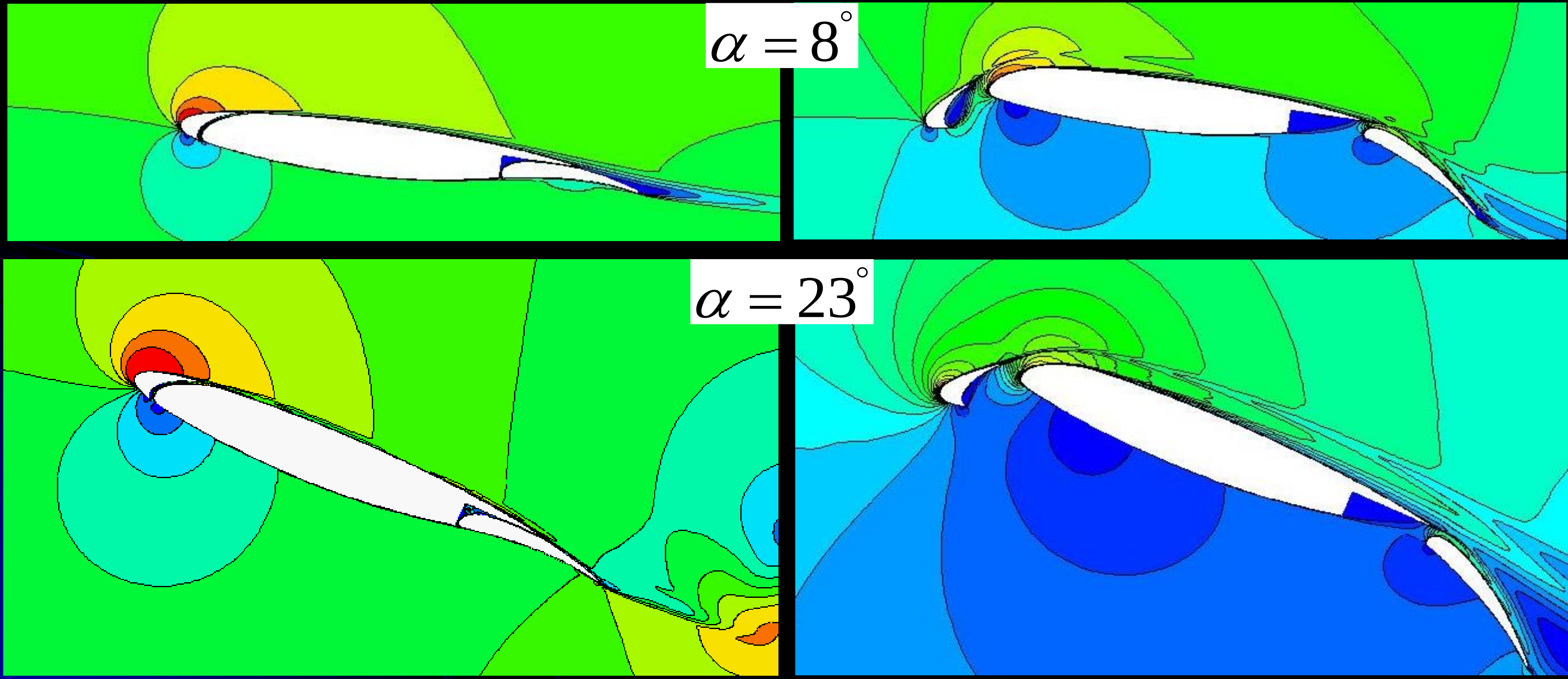
Changing
the drag coefficient

0.025

Aerodynamics of aircraft elements



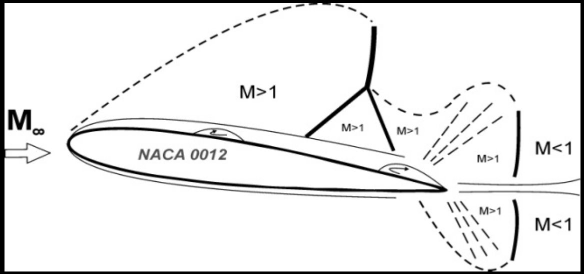
Three-element 30P30N airfoil



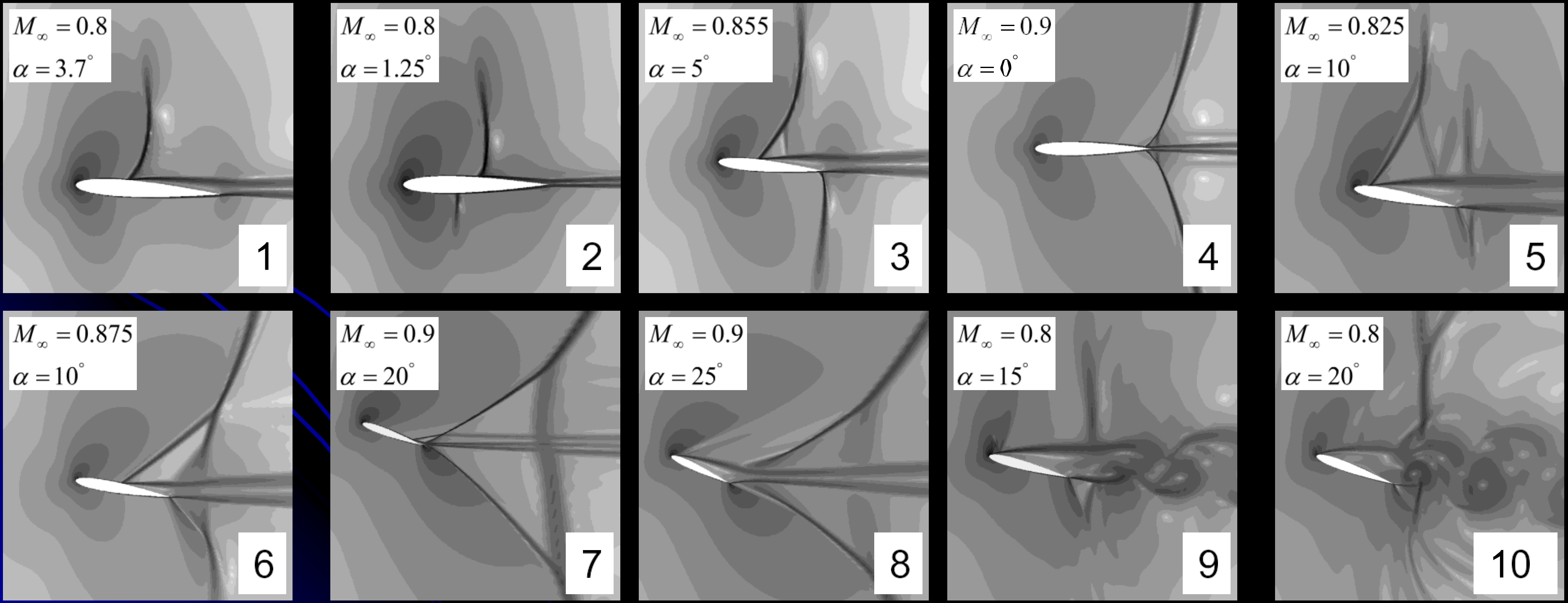
Cruising configuration

Takeoff and landing configuration

Transonic aerodynamics

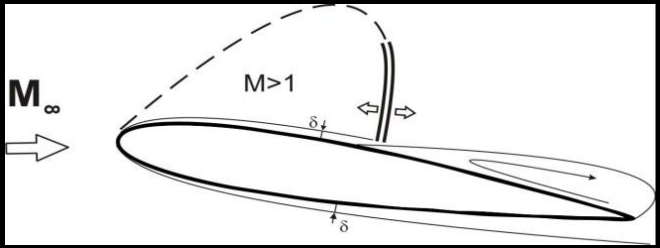


Study of flow regimes

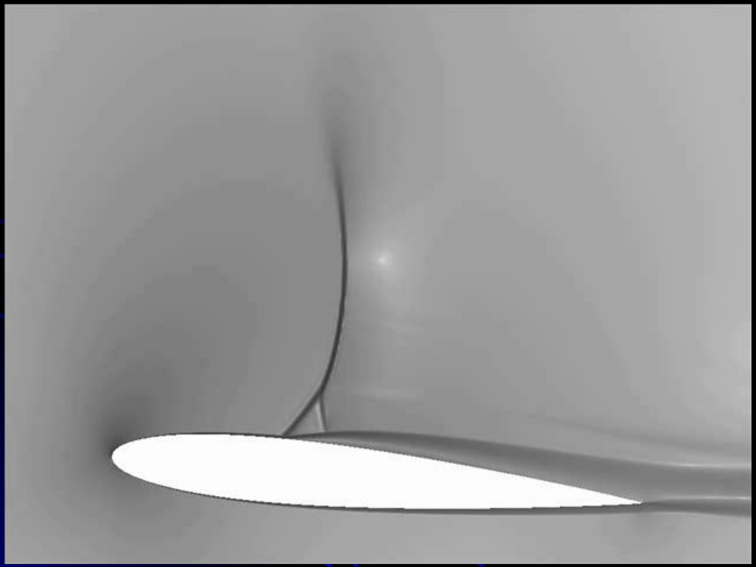


Numerical “Schlieren photographs" for flow around NACA 0012 airfoil

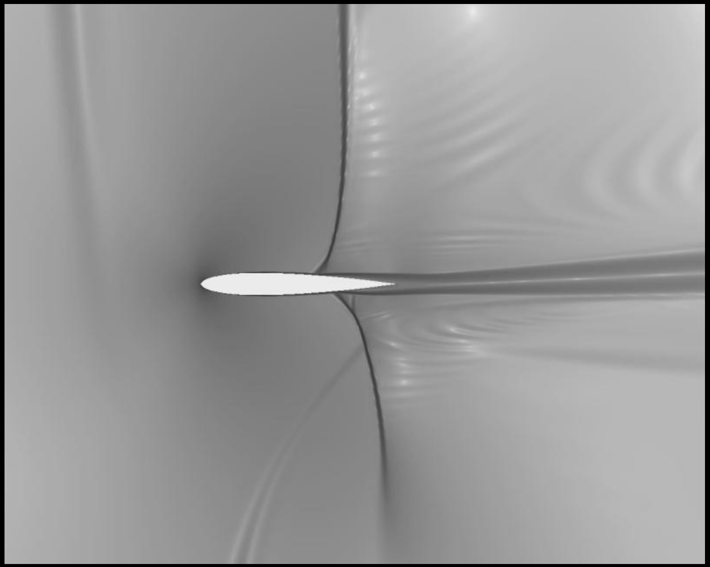
Transonic aerodynamics



Self-induced shock waves oscillations

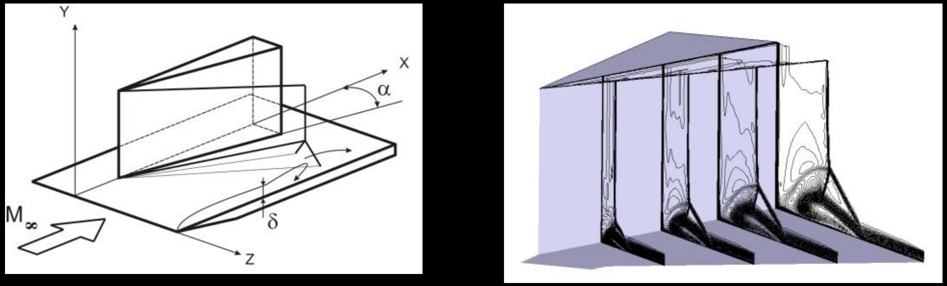


One shock wave self-induced oscillations
M_∞=0.725 α=5.0^0

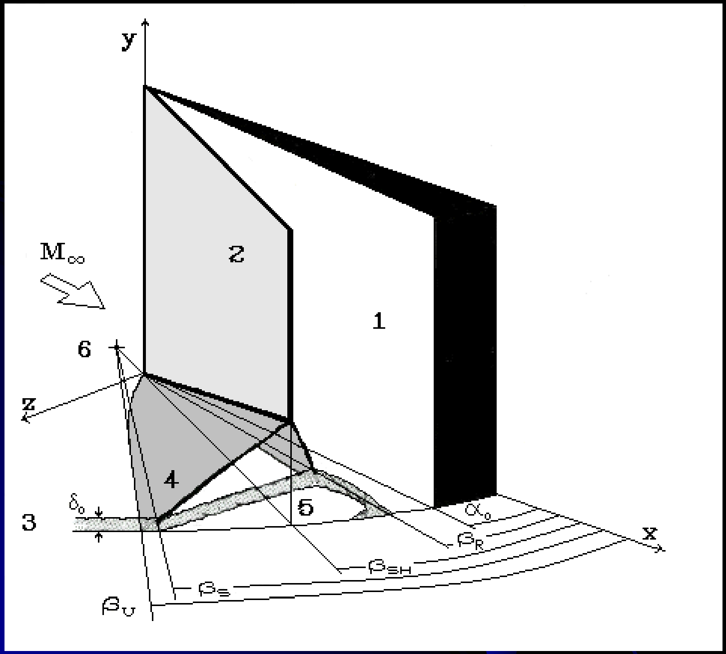


Two shock waves self-induced oscillations
M_∞=0.855 α=0.0^0

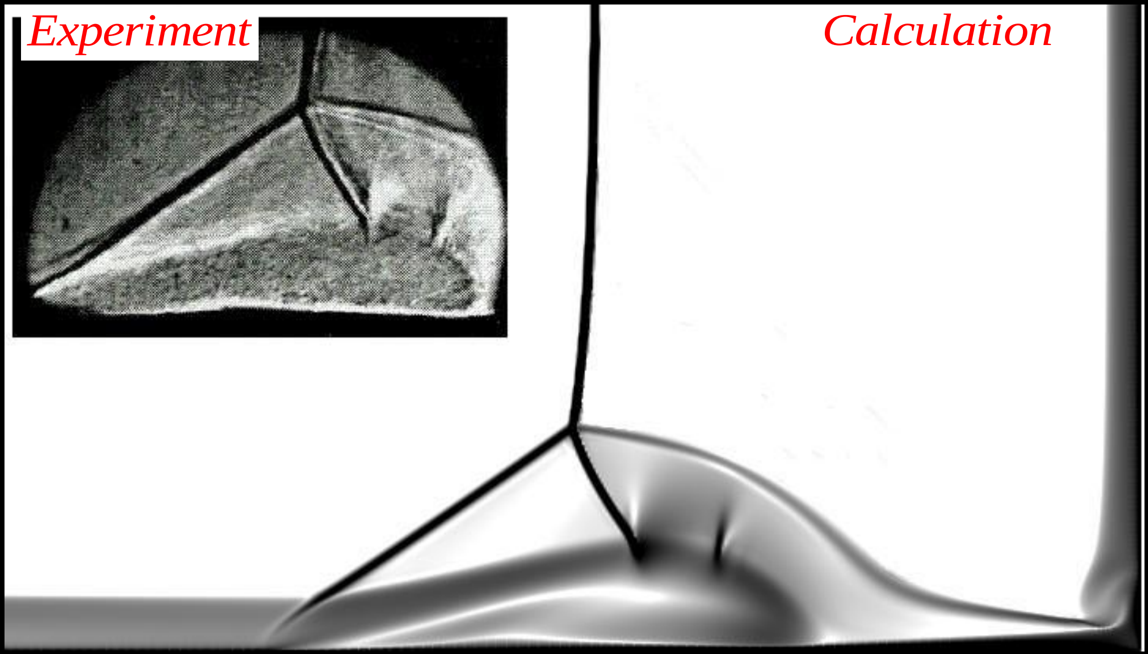
Supersonic aerodynamics of aircraft, rocket elements



Sharp fin mounted on a plate

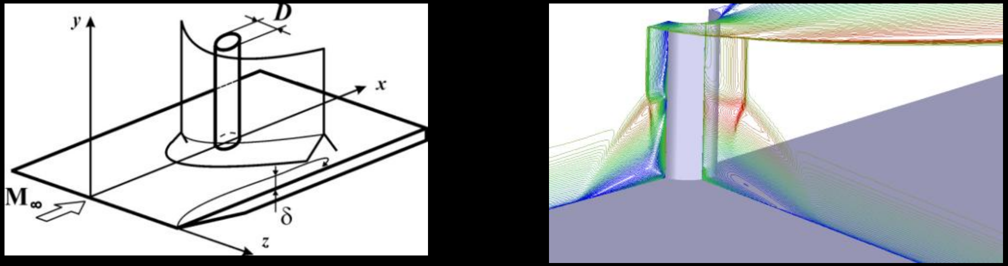


General structure of 3D interaction

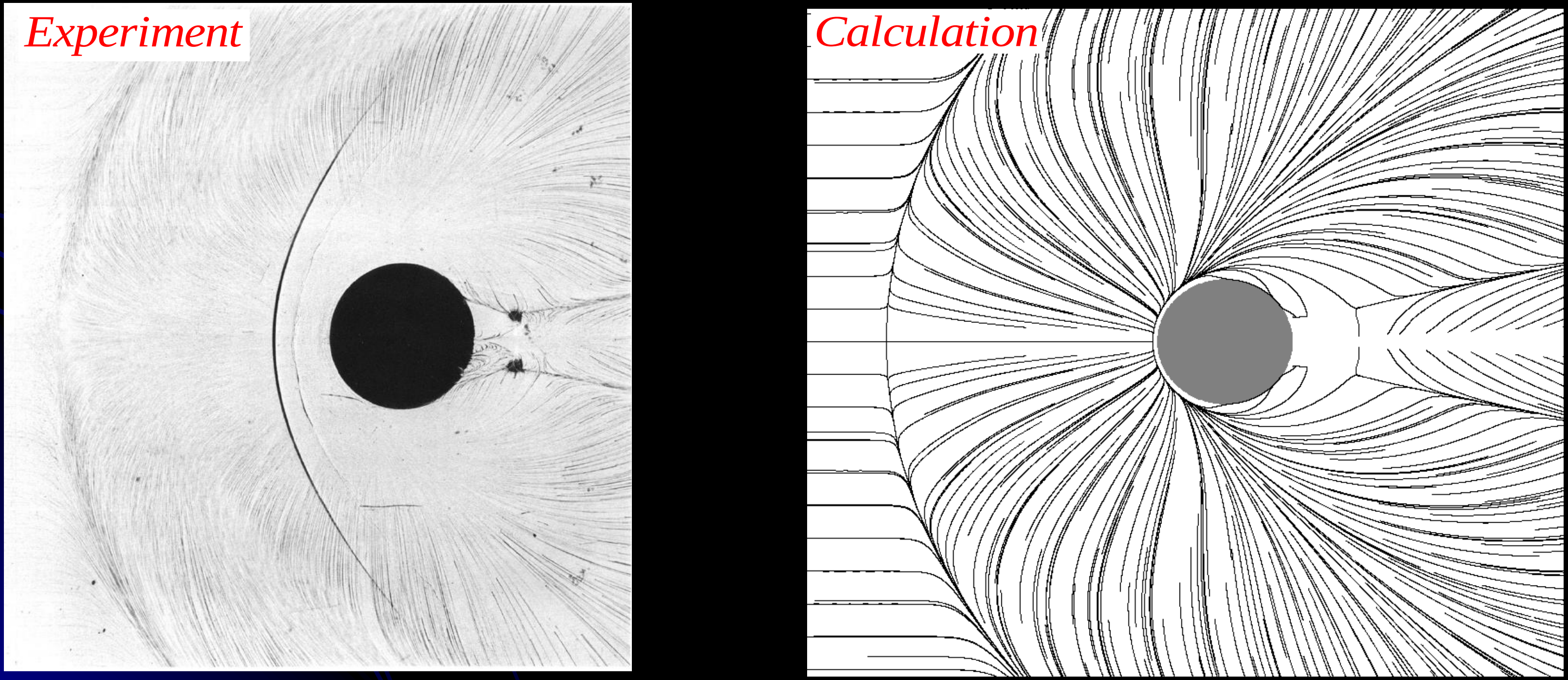


Schlieren photography

Supersonic aerodynamics of aircraft, rocket elements



Cylinder mounted on the plate

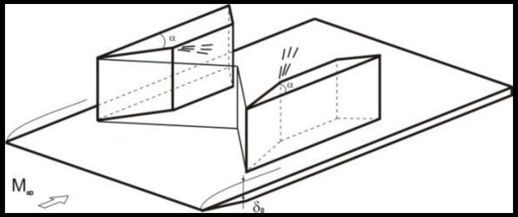


Experiment

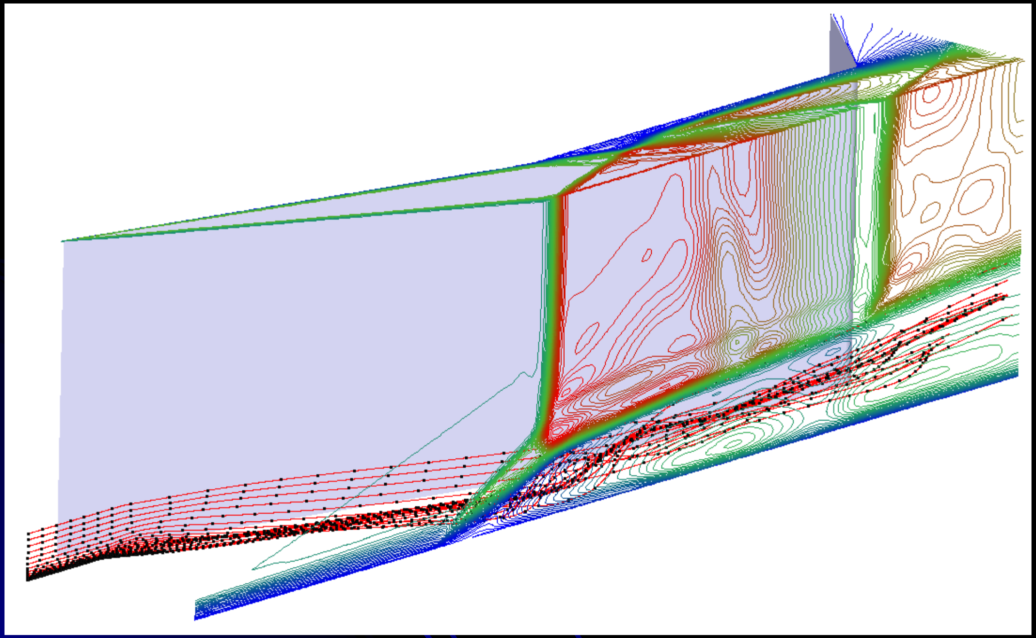
Calculation

Distribution of limiting streamlines on the plate surface

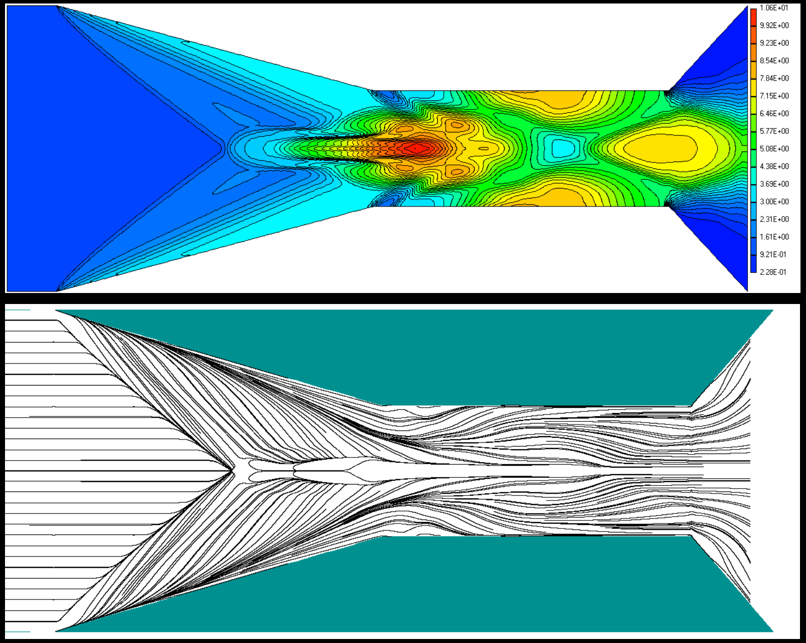
Supersonic aerodynamics of aircraft and rocket elements



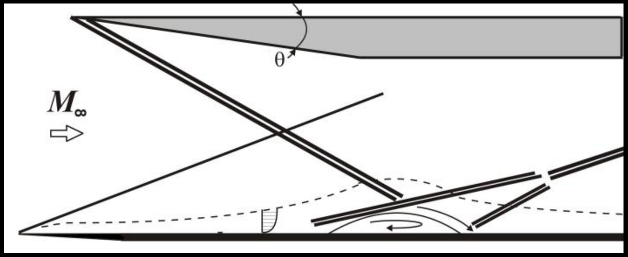
3D air inlet



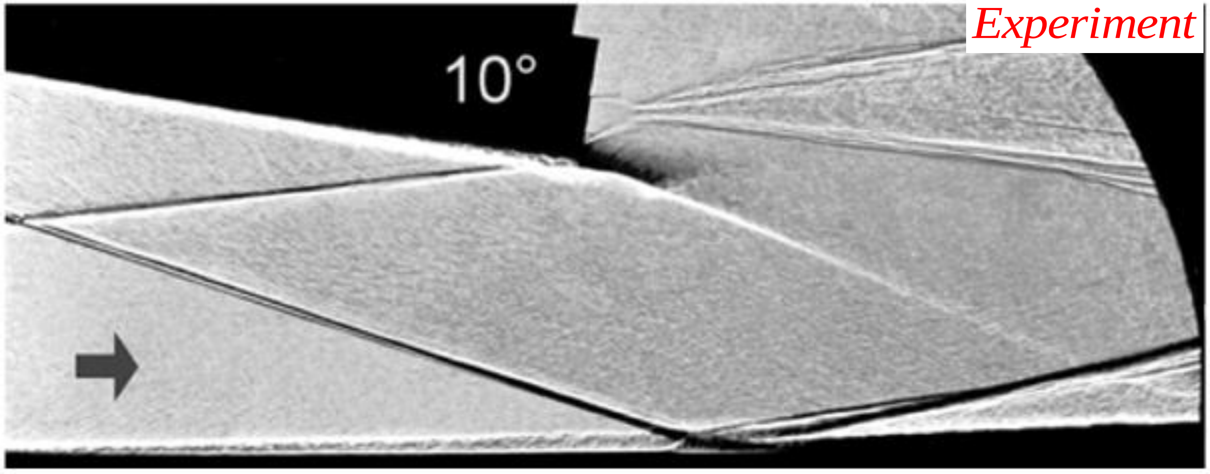
Distribution of local Mach numbers and spatial streamlines in the air inlet

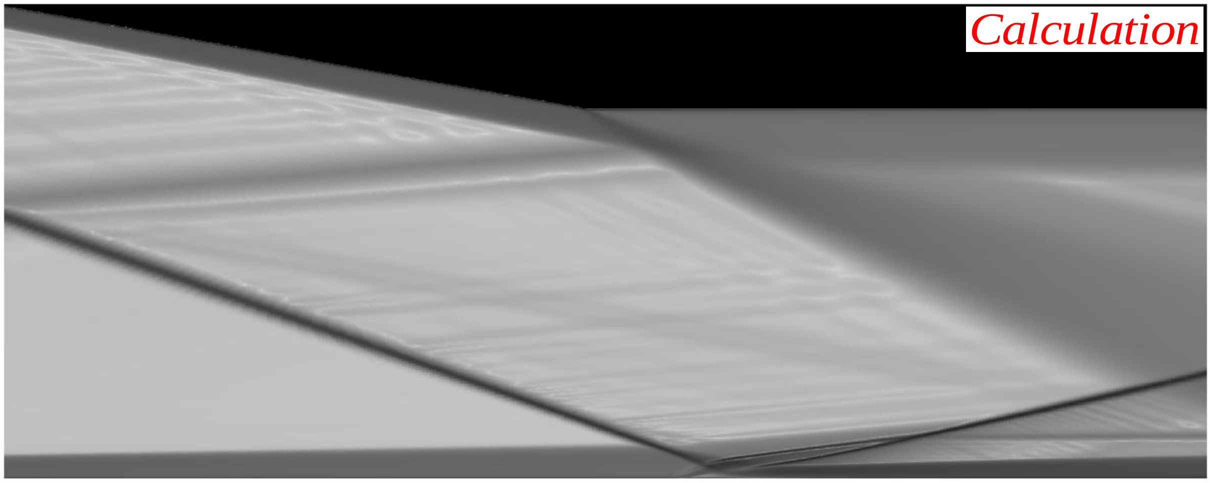


Pressure distribution and limiting streamlines on the lower surface of the air inlet

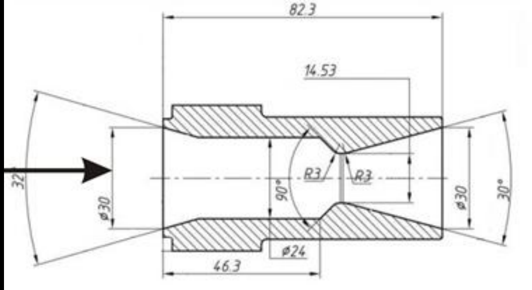


Hypersonic aerodynamics of aircraft, rocket elements

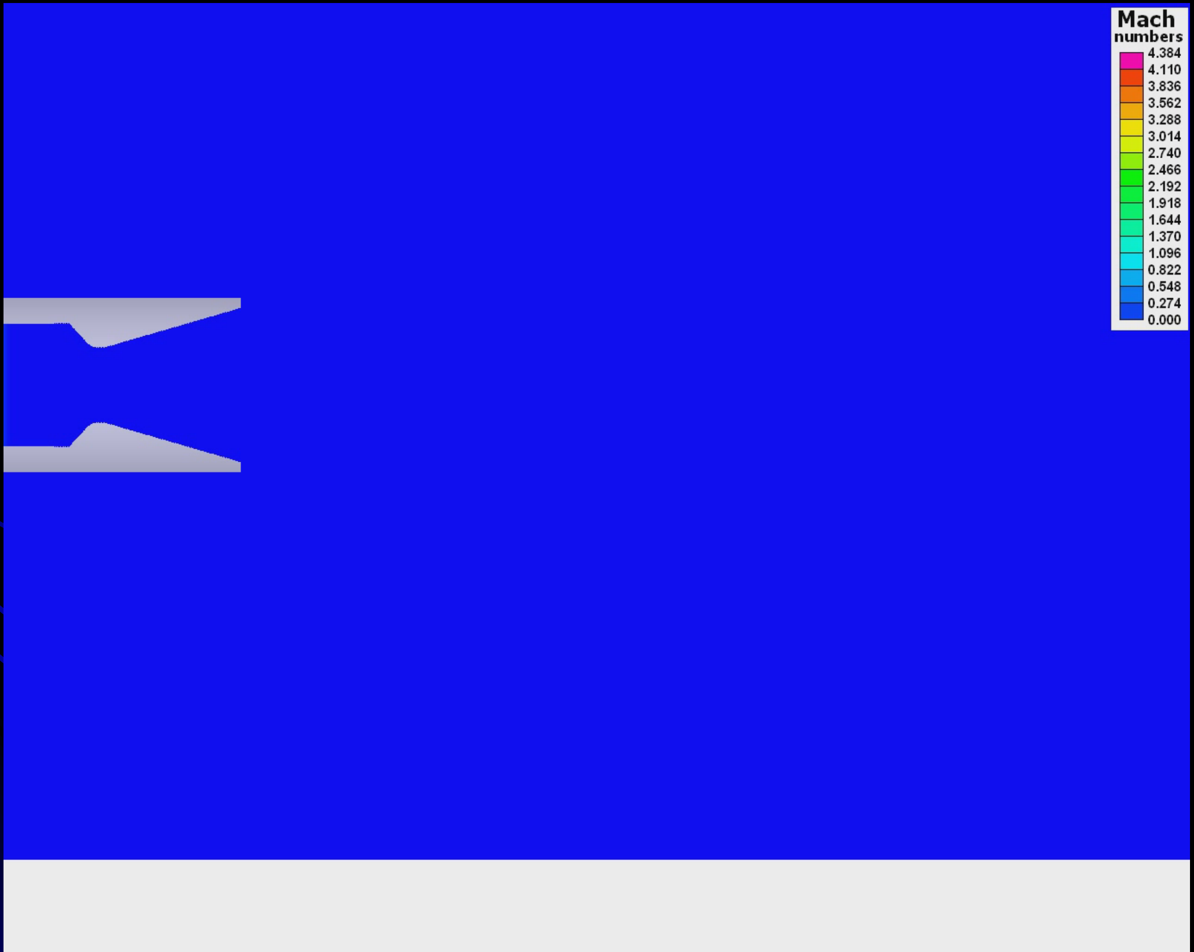




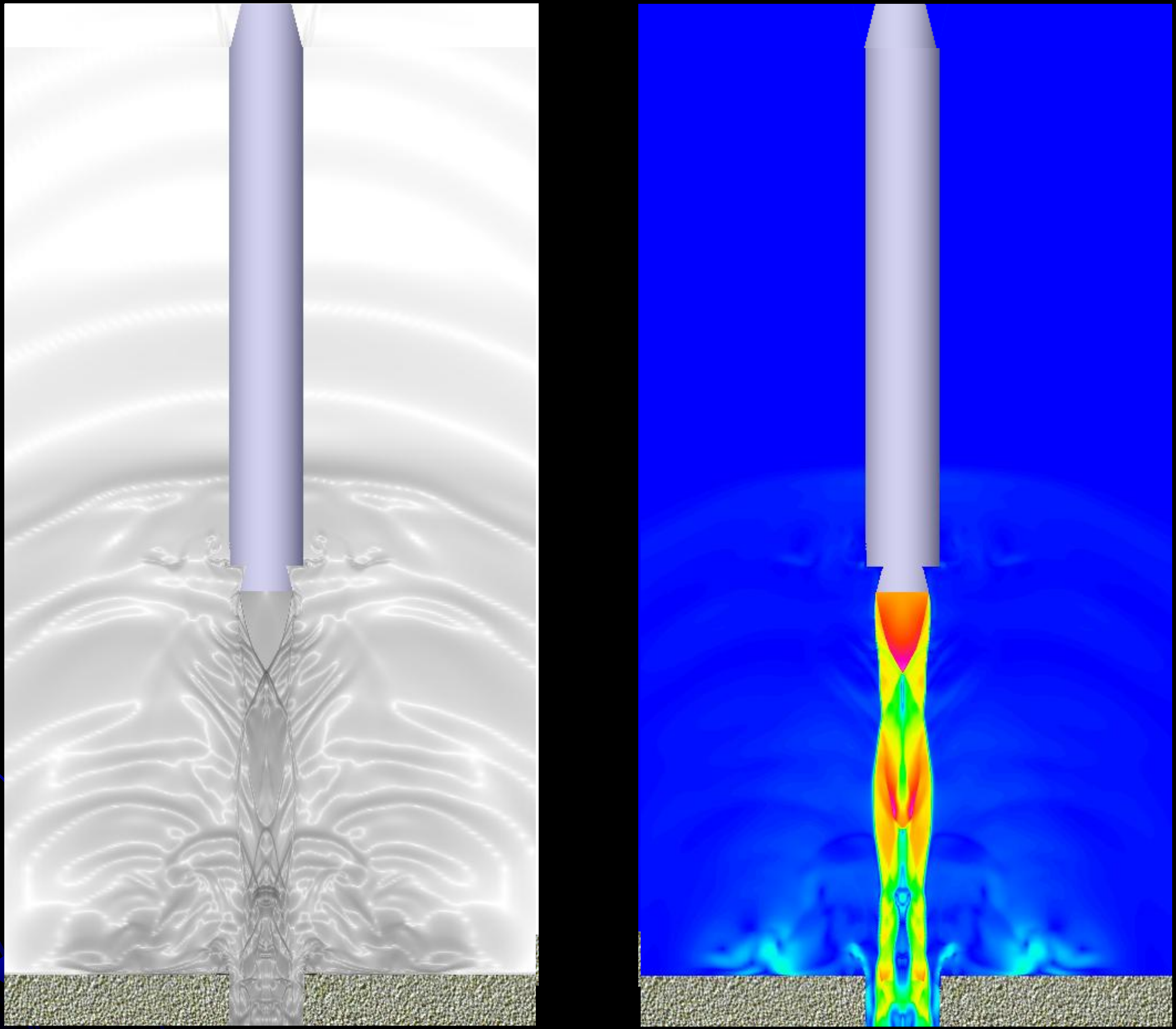
2D hypersonic shock wave/turbulent boundary layer interaction



Origin and development of axisymmetric supersonic air jet



Numerical simulation of gas dynamics and acoustics during rocket launch



Thank you for attention!